The mod-2 Hopf Rings for *ko* and *tmf*

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Outline

Key ideas

Things well known

Things less well known

Hopf rings

Dieudonné rings

Brown-Gitler Spectra

Adams spectral sequence

Adams spectral sequence for ko

Dieudonné ring for ko

Margolis homology

Ext for $k(r)$

Dieudonné and Hopf ring results for $k(r)$

References

Key ideas

Let E be a ring spectrum and $\underline{E}_k = \Omega^\infty \Sigma^k E$ the k^{th} space in its Ω-spectrum.

- A Hopf ring $H_*(\underline{E}_*)$ is
  - a ring object in the category of coalgebras over $\mathbb{F}$
  - dual to the algebra of unstable cohomology operations $E^*(-) \to H^*(-)$
- Hopf rings and Dieudonné rings are equivalent

$$H_n(\underline{E}_k) \equiv D_n(H.(\underline{E}_k))$$

- There is a close connection between Dieudonné rings and Brown-Gitler spectra:

$$E_{n-k}(B(n)) \twoheadrightarrow D_n(H.(\underline{E}_k))$$

- The Adams spectral sequence can be used to compute $E_*(B(n))$:

$$H_*(E) \otimes H_*(B(n)) \otimes \Lambda \implies E_*(B(n))$$

Key ideas

- There is a cofiber sequence

$$B(2n-2) \to B(2n) \xrightarrow{v} \Sigma^n B(n)$$

that creates a short exact sequence in mod 2 homology and a long exact sequence in Ext (i.e., Adams sseq E_2-terms) and can be used for induction

- The unit maps of ring spectra

$$\mathbb{S} \to \text{tmf} \to \text{ko} \to \text{H}\mathbb{Z} \to \text{H}\mathbb{F}$$

can be used to relate Hopf rings

$$H_n(QS^k) \to H_n(\underline{\text{tmf}}_k) \to H_n(\underline{\text{ko}}_k) \to H_n(\underline{\text{H}\mathbb{Z}}_k) \to H_n(\underline{\text{H}\mathbb{F}}_k)$$

and their associated Dieudonné rings and Adams sseq's

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The Hopf ring for the sphere



Theorem (Turner, Wilson, Eccles)

Let $E_i = Q^i[1] \in \mathrm{H}\mathbb{F}_i(QS^0)$ *and define* $E(s) = \sum_{i\geq 0} E_i s^i$. *Then the Hopf ring for the sphere has*

$$\mathrm{H}\mathbb{F}_*(QS^0) = \frac{\mathbb{F}[E_{i_0} \circ \cdots \circ E_{2^{n-1}(i_0+\cdots i_{n-1})} | i_0 > 0, n \geq 1] \otimes \mathbb{F}[\mathbb{Z}]}{(E_0 = [2], E(s) \circ E(t) = E(s+t) \circ E(t))}$$

and for $k > 0$

$$\mathrm{H}\mathbb{F}_*(QS^k) = \frac{\mathbb{F}[e^{\circ k} \circ E_{i_0} \circ \cdots \circ E_{2^{n-1}(i_0+\cdots i_{n-1})} | i_0 > k, n \geq 1] \otimes \mathbb{F}[e^{\circ k}]}{(E_0 = [2], E(s) \circ E(t) = E(s+t) \circ E(t)), e^{\circ i} \circ E_i = (e^{\circ i})^{*2})}$$

Mod 2 Hopf ring for $\mathbb{HF}$

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Theorem (Wilson)

As Hopf algebras over $\mathbb{F}_2$ with addition $+$ and multiplication $*$, $H_*(\underline{\mathbb{HF}}_0) = \mathbb{F}_2[\mathbb{F}_2]$ and

$$H_*(\underline{\mathbb{HF}}_k) = E(x_{i_1} \circ \cdots \circ x_{i_k} \mid 0 \leq i_1 \leq \cdots \leq i_k).$$

Further, $*$-Ind$(H_*(\underline{\mathbb{HF}}_*)) = Sym(x_i \mid i \geq 0$, the bigraded symmetric algebra over $\mathbb{F}_2$ with addition $+$, multiplication $\circ$, and generators $x_i \in H_{2^i}(\underline{\mathbb{HF}}_1)$.

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The Hopf ring for ko

Conjecture (Pearson)

The homology of $\underline{\mathrm{ko}}_0 = \mathbb{Z} \times BO$*, the zero space of the connective real K-theory spectrum, is*

$$H_*(\underline{\mathrm{ko}}_0) = \frac{H_*(QS^0)}{(N(E_i) \mid \alpha(i) > 1)}$$

where $N(E_i)$ *is the Newton polynomial in* $E_0, \ldots, E_i$ *and* α *is the number of ones in the* 2*-adic valuation of* i*.*

Note: I have calculated the Dieudonné ring for ko and published the results. Translating those results to the Hopf ring for ko should yield a proof of this conjecture.

The Hopf ring for tmf

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Conjecture (Pearson)

The homology of $\underline{\mathrm{tmf}}_0$*, the zero space of the connective topological modular forms spectrum, is*

$$H_*(\underline{\mathrm{tmf}}_0) = \frac{H_*(QS^0)}{(N(E_i) \mid \alpha(i) > 2)}.$$

for $0 \leq * \leq 59$ *and possibly beyond.*

Steve Wilson has shown this to be false in $\mathrm{H}\mathbb{F}_{24}(\underline{\mathrm{tmf}}_0)$ where there is likely an additional relation coming from $\mathrm{H}\mathbb{F}_0(\underline{\mathrm{tmf}}_{-24}) = \mathbb{F}_2[\pi_{24}^S(\mathrm{tmf})]$ via homology suspension. I do think that $\mathrm{H}\mathbb{F}_*(QS^0) \to \mathrm{H}\mathbb{F}_*(\underline{\mathrm{tmf}}_0)$ is surjective, though, because I'm wildly optimistic.

Summary

I conjecture that for tmf and ko (and possibly other connective ring spectra E), their mod 2 Hopf ring $H_*(\underline{E}_*)$ is

- part quotient of $H_*(QS^*)$
- part subobject of $H_*(\underline{\mathbb{HF}}_*)$
- and that these parts get glued together by relations

Visualizing Hopf rings

Some incomplete plots of Hopf rings:

- Hopf ring for the Sphere
- Hopf ring for tmf
- Hopf ring for ko
- Hopf ring for mod 2 Eilenberg-Mac Lane spectrum

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Coalgebraic groups

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Definition

A Hopf algebra is a coalgebraic group, i.e., a group object in a category of coalgebras.

Example

The collection $(H_*(\underline{E}_k), \epsilon, \psi, \eta, *, \chi)$ comprises a Hopf algebra, or coalgebraic group, over $\mathbb{F}$ with

- cycle addition $+$
- loop multiplication $*$
- loop reversal (conjugation) χ
- comultiplication ψ
- unit η
- augmentation ϵ

Coalgebraic rings

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Definition

A Hopf ring is a coalgebraic ring, i.e., a ring object in a category of coalgebras.

Example

When E is a ring spectrum, the collection $(H_*(\underline{E}_*), \epsilon, \psi, \eta, *, \chi, \circ)$ comprises a Hopf ring, or coalgebraic ring, over $\mathbb{F}$ with

- cycle addition $+$
- loop multiplication $*$
- loop reversal (conjugation) χ
- comultiplication ψ
- unit η
- augmentation ϵ
- circle product $\circ$

Hopf rings

There are three products in a Hopf ring

- **Star product:** Each space $\underline{E}_k$ in the Ω-spectrum associated to E is a pointed infinite loop space with a loop multiplication that induces the star product.

$$(\textit{loop mult.}) \qquad * : H_m(\underline{E}_k) \otimes H_n(\underline{E}_k) \to H_{m+n}(\underline{E}_k)$$

- **Circle product:** The ring multiplication $\circ : \underline{E}_j \times \underline{E}_k \to \underline{E}_{j+k}$ of the spectrum E induces the circle product.

$$(\textit{circle prod.}) \qquad \circ : H_m(\underline{E}_j) \otimes H_n(\underline{E}_k) \to H_{m+n}(\underline{E}_{j+k})$$

- **Addition:** Each $H_n(\underline{E}_k)$ is a group under addition of cycles.

$$(\textit{cycle addition}) \qquad + : H_n(\underline{E}_k) \otimes H_n(\underline{E}_k) \to H_n(\underline{E}_k)$$

Hopf rings

- **Multiplicative identity:** The circle product multiplicative identity $[1] \in H_0(\underline{E}_0)$ is the image of the non base-point generator under the inclusion $S^0 \to \underline{E}_0$
- **Homology suspension:** The homology suspension class $e \in H_1(\underline{E}_1)$ is the image of the generator under $S^1 \to \underline{E}_1$ and is primitive. The homology suspension of x is $e \circ x$, where
$$e \circ (-) : H_n(\underline{E}_k) \to H_{n+1}(\underline{E}_{k+1}).$$
- **Conjugation:** The map $\chi : \underline{E}_k \to \underline{E}_k$ that reverses loop orientation induces conjugation
$$\chi : H_n(\underline{E}_k) \to H_n(\underline{E}_k).$$

Hopf algebras

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Definition

The Frobenius and Verschiebung in a Hopf algebra $H_*(\underline{E}_k)$ of finite type are

$$\begin{array}{lrcccc} (Frobenius) & F & : & H_n(\underline{E}_k) & \to & H_{2n}(\underline{E}_k) \\ & & & y & \mapsto & y^{*2} \end{array}$$

$$\begin{array}{lrcccc} (Verschiebung) & V & : & H_{2n}(\underline{E}_k) & \to & H_n(\underline{E}_k) \\ & & & y & \mapsto & ax \end{array}$$

where $\psi(y) = a(x \otimes x) +$ other terms, and V is zero on $H_{2n+1}(\underline{E}_k)$.

Hopf rings



Theorem

In a Hopf ring $H_*(\underline{E}_*)$, the Frobenius and Verschiebung satisfy the Frobenius reciprocity relations

$$\begin{aligned} V(x \circ y) &= V(x) \circ V(y) \\ F(x) \circ y &= F(x \circ V(y)) \\ x \circ F(y) &= F(V(x) \circ y) \end{aligned}$$

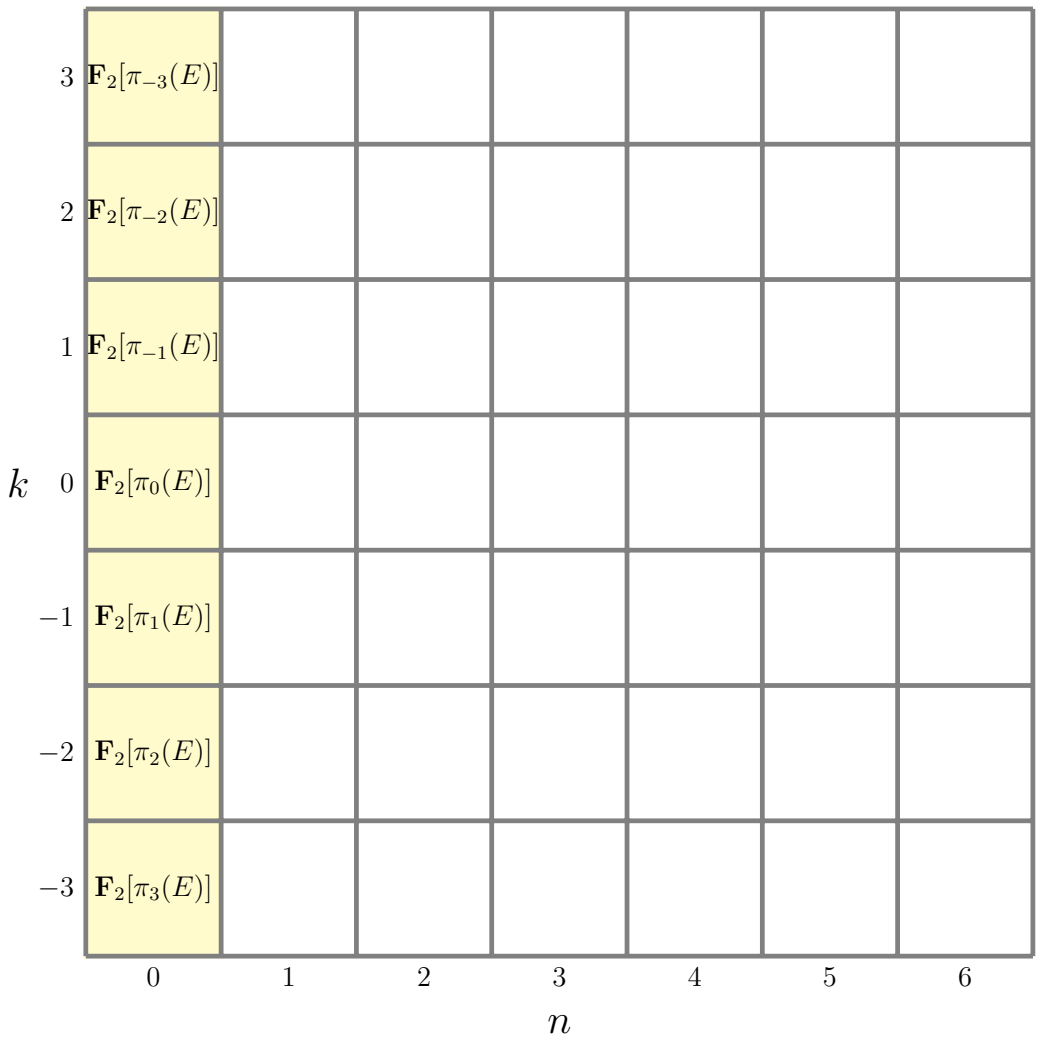
Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$

k \ n	0	1	2	3	4	5	6
3	$H_0(\underline{E}_3)$	$H_1(\underline{E}_3)$	$H_2(\underline{E}_3)$	$H_3(\underline{E}_3)$	$H_4(\underline{E}_3)$	$H_5(\underline{E}_3)$	$H_6(\underline{E}_3)$
2	$H_0(\underline{E}_2)$	$H_1(\underline{E}_2)$	$H_2(\underline{E}_2)$	$H_3(\underline{E}_2)$	$H_4(\underline{E}_2)$	$H_5(\underline{E}_2)$	$H_6(\underline{E}_2)$
1	$H_0(\underline{E}_1)$	$H_1(\underline{E}_1)$	$H_2(\underline{E}_1)$	$H_3(\underline{E}_1)$	$H_4(\underline{E}_1)$	$H_5(\underline{E}_1)$	$H_6(\underline{E}_1)$
0	$H_0(\underline{E}_0)$	$H_1(\underline{E}_0)$	$H_2(\underline{E}_0)$	$H_3(\underline{E}_0)$	$H_4(\underline{E}_0)$	$H_5(\underline{E}_0)$	$H_6(\underline{E}_0)$
-1	$H_0(\underline{E}_{-1})$	$H_1(\underline{E}_{-1})$	$H_2(\underline{E}_{-1})$	$H_3(\underline{E}_{-1})$	$H_4(\underline{E}_{-1})$	$H_5(\underline{E}_{-1})$	$H_6(\underline{E}_{-1})$
-2	$H_0(\underline{E}_{-2})$	$H_1(\underline{E}_{-2})$	$H_2(\underline{E}_{-2})$	$H_3(\underline{E}_{-2})$	$H_4(\underline{E}_{-2})$	$H_5(\underline{E}_{-2})$	$H_6(\underline{E}_{-2})$
-3	$H_0(\underline{E}_{-3})$	$H_1(\underline{E}_{-3})$	$H_2(\underline{E}_{-3})$	$H_3(\underline{E}_{-3})$	$H_4(\underline{E}_{-3})$	$H_5(\underline{E}_{-3})$	$H_6(\underline{E}_{-3})$

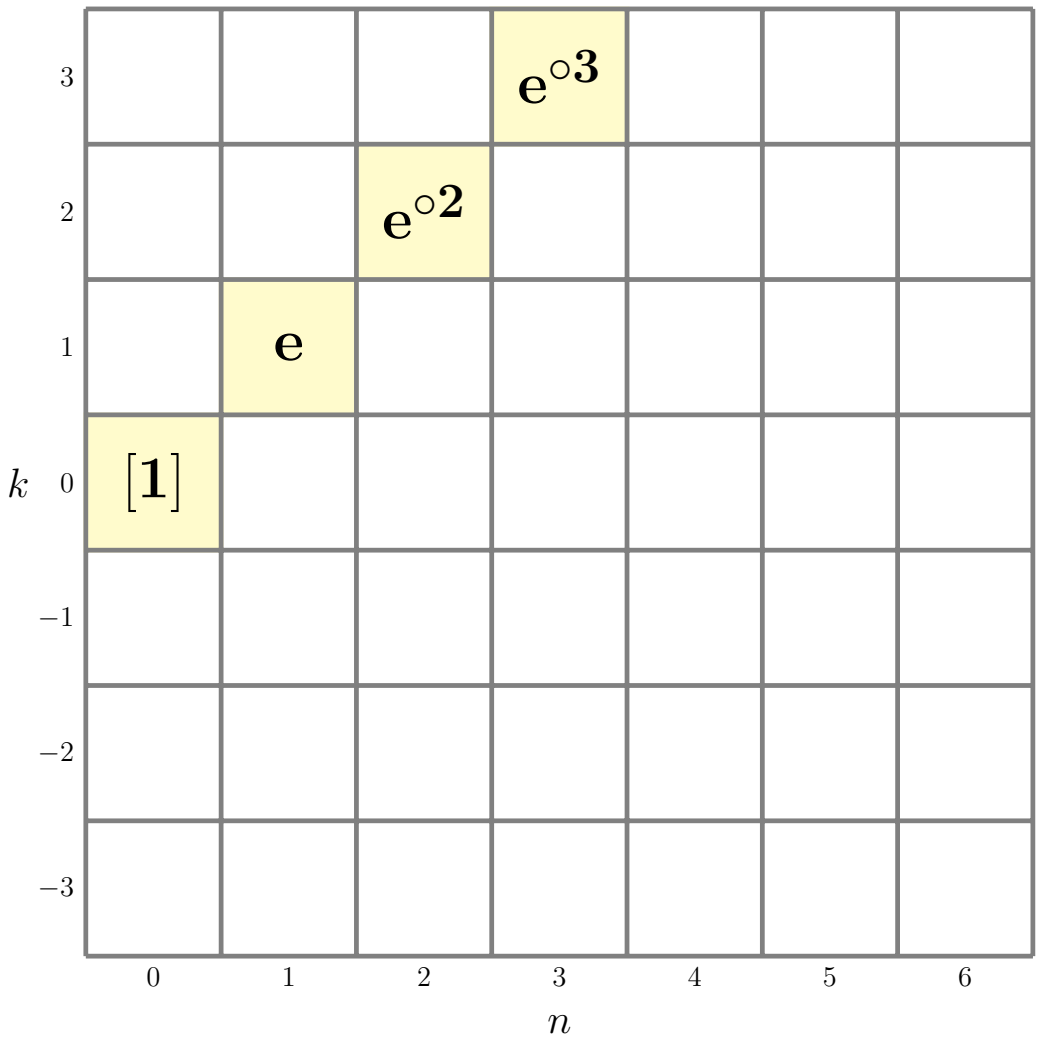
Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$



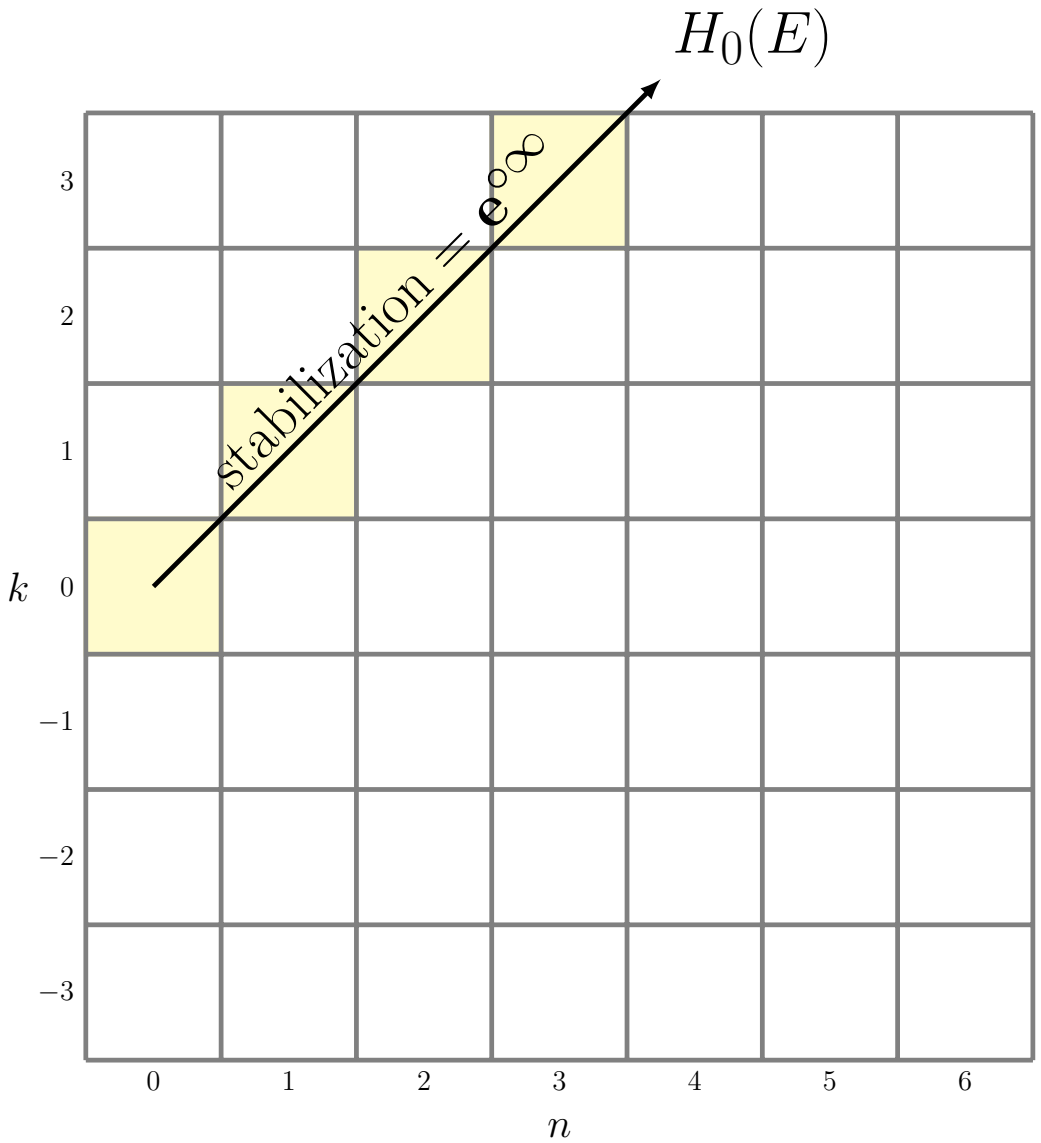
Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$



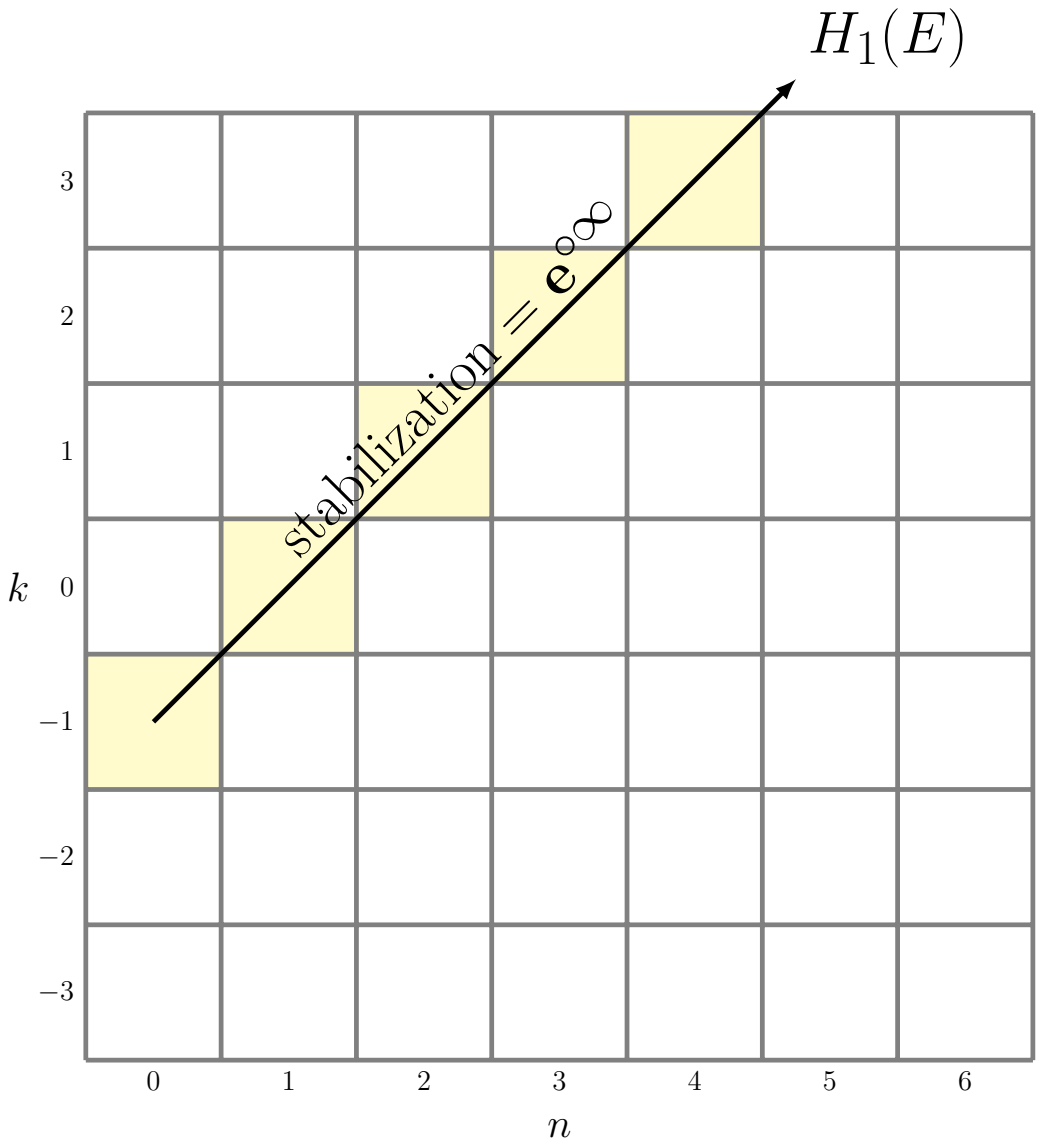
Hopf ring structure

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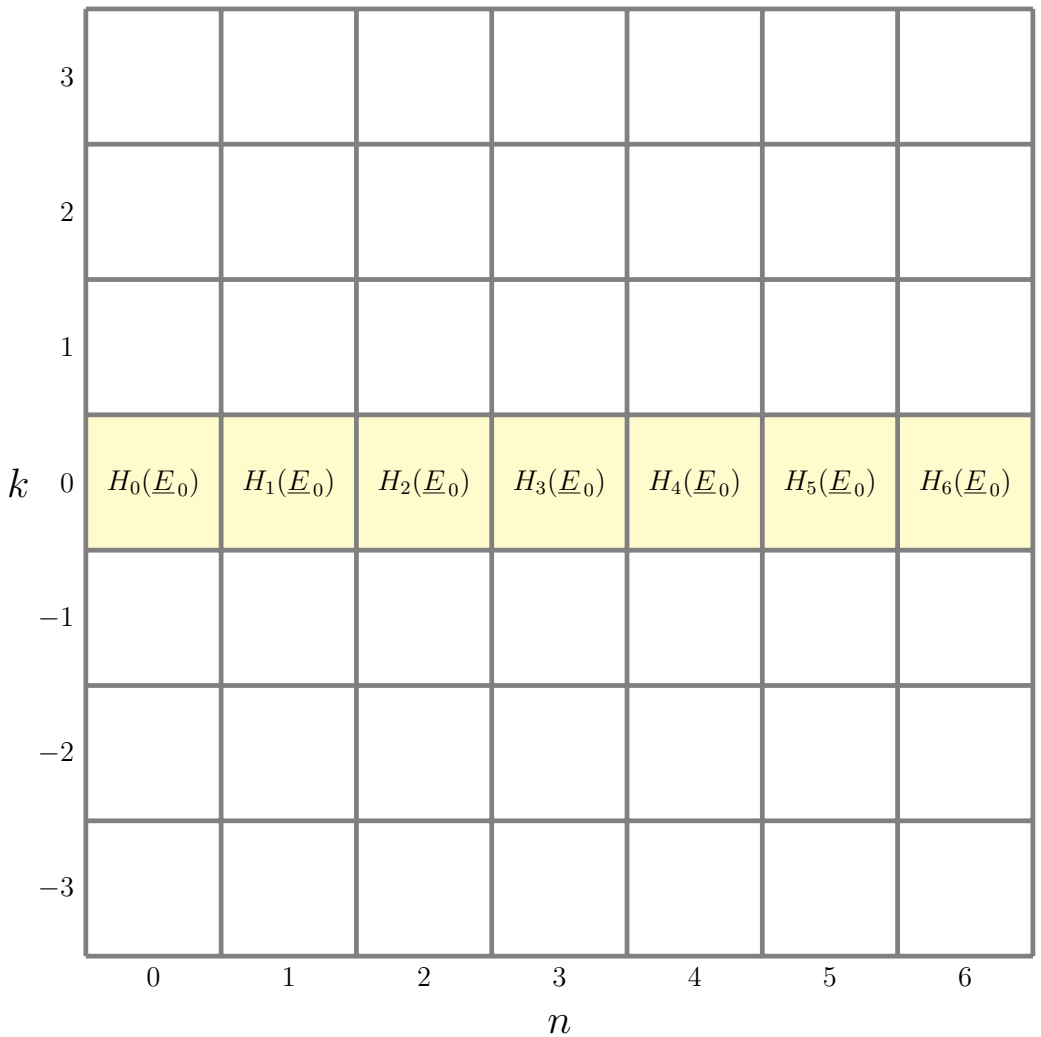
Hopf ring structure

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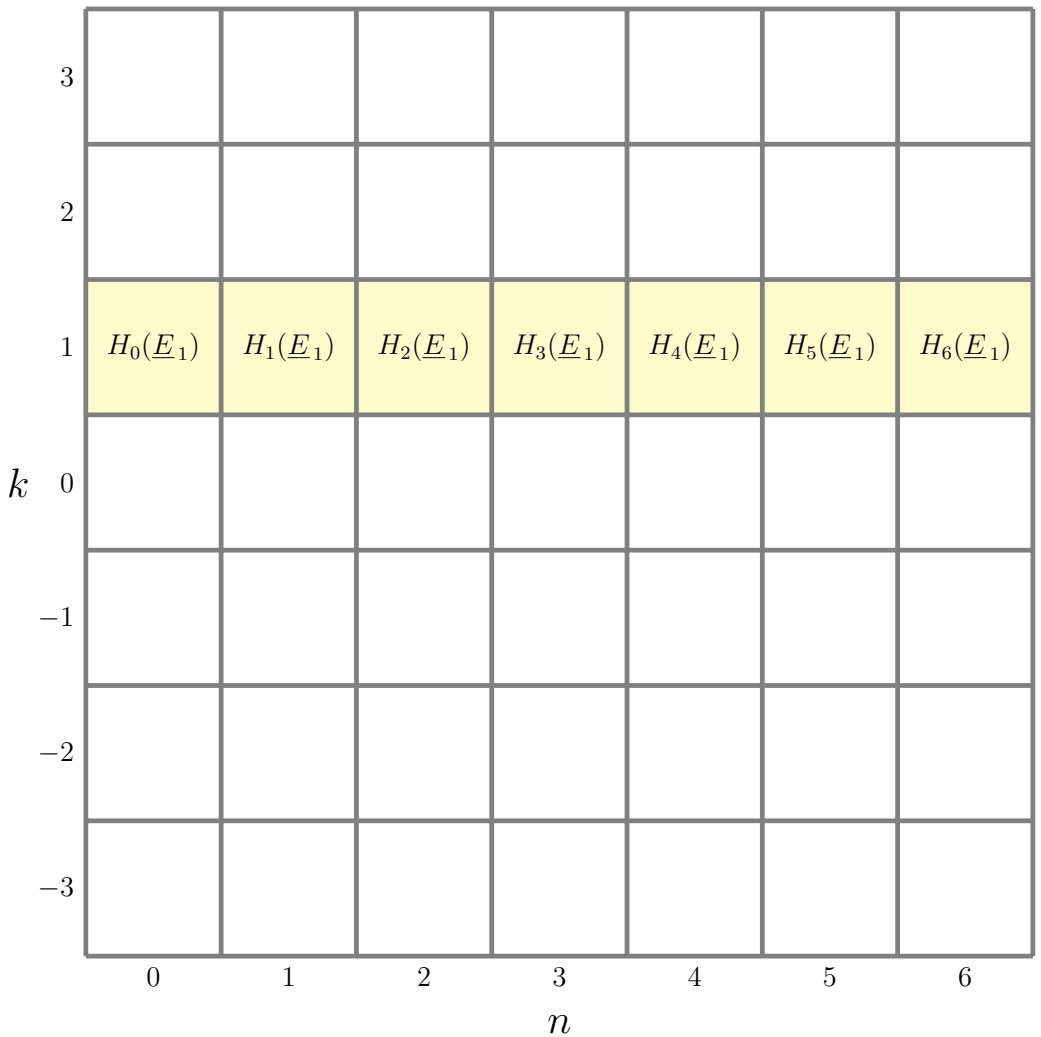
Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$



Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$



Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$

If E is 2-periodic and non-connective:

k \ n	0	1	2	3	4	5	6
3	$H_0(\underline{E}_3)$	$H_1(\underline{E}_3)$	$H_2(\underline{E}_3)$	$H_3(\underline{E}_3)$	$H_4(\underline{E}_3)$	$H_5(\underline{E}_3)$	$H_6(\underline{E}_3)$
2	$H_0(\underline{E}_2)$	$H_1(\underline{E}_2)$	$H_2(\underline{E}_2)$	$H_3(\underline{E}_2)$	$H_4(\underline{E}_2)$	$H_5(\underline{E}_2)$	$H_6(\underline{E}_2)$
1	$H_0(\underline{E}_1)$	$H_1(\underline{E}_1)$	$H_2(\underline{E}_1)$	$H_3(\underline{E}_1)$	$H_4(\underline{E}_1)$	$H_5(\underline{E}_1)$	$H_6(\underline{E}_1)$
0	$H_0(\underline{E}_0)$	$H_1(\underline{E}_0)$	$H_2(\underline{E}_0)$	$H_3(\underline{E}_0)$	$H_4(\underline{E}_0)$	$H_5(\underline{E}_0)$	$H_6(\underline{E}_0)$
-1	$H_0(\underline{E}_{-1})$	$H_1(\underline{E}_{-1})$	$H_2(\underline{E}_{-1})$	$H_3(\underline{E}_{-1})$	$H_4(\underline{E}_{-1})$	$H_5(\underline{E}_{-1})$	$H_6(\underline{E}_{-1})$
-2	$H_0(\underline{E}_{-2})$	$H_1(\underline{E}_{-2})$	$H_2(\underline{E}_{-2})$	$H_3(\underline{E}_{-2})$	$H_4(\underline{E}_{-2})$	$H_5(\underline{E}_{-2})$	$H_6(\underline{E}_{-2})$
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Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$

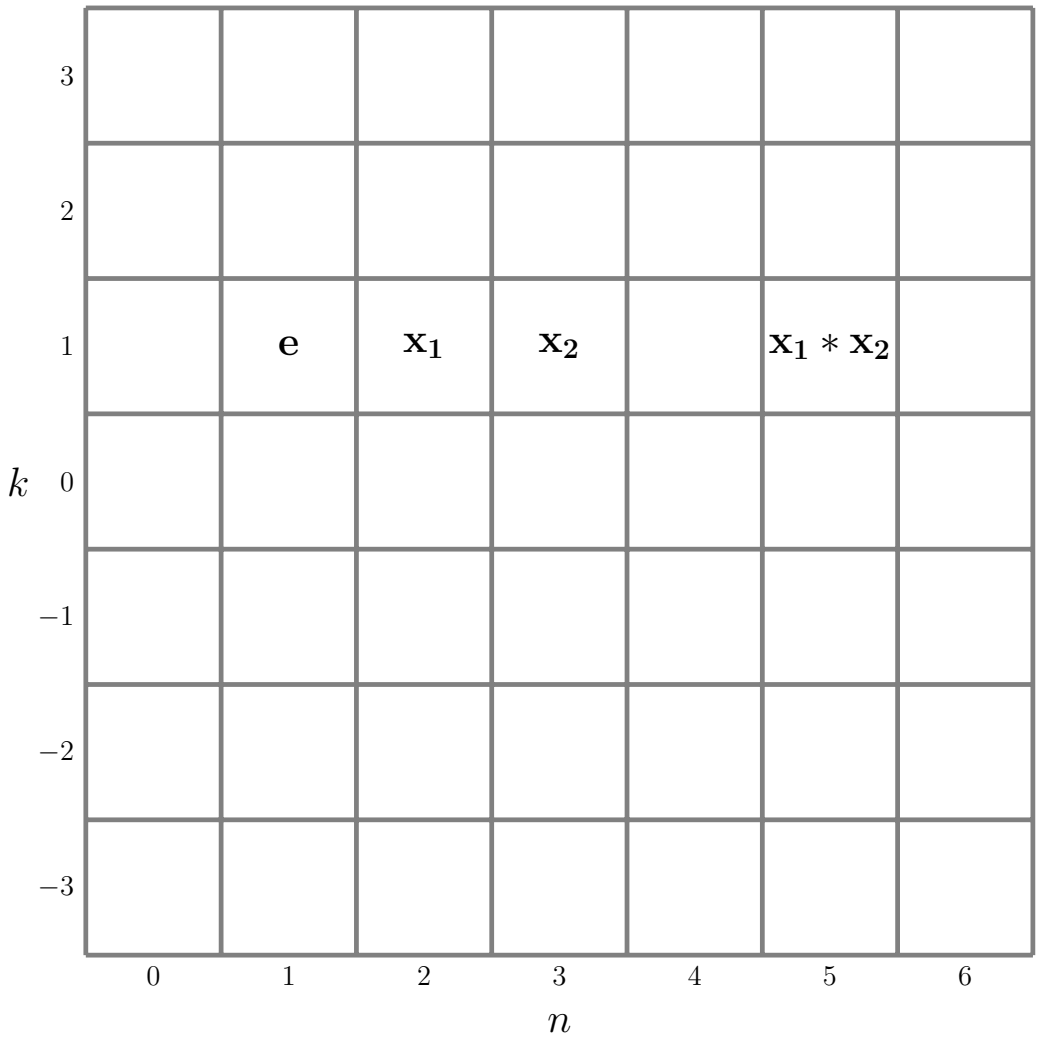
If E is 2-periodic and connective:

k \ n	0	1	2	3	4	5	6
3	$H_0(\underline{E}_3)$	$H_1(\underline{E}_3)$	$H_2(\underline{E}_3)$	$H_3(\underline{E}_3)$	$H_4(\underline{E}_3)$	$H_5(\underline{E}_3)$	$H_6(\underline{E}_3)$
2	$H_0(\underline{E}_2)$	$H_1(\underline{E}_2)$	$H_2(\underline{E}_2)$	$H_3(\underline{E}_2)$	$H_4(\underline{E}_2)$	$H_5(\underline{E}_2)$	$H_6(\underline{E}_2)$
1	$H_0(\underline{E}_1)$	$H_1(\underline{E}_1)$	$H_2(\underline{E}_1)$	$H_3(\underline{E}_1)$	$H_4(\underline{E}_1)$	$H_5(\underline{E}_1)$	$H_6(\underline{E}_1)$
0	$H_0(\underline{E}_0)$	$H_1(\underline{E}_0)$	$H_2(\underline{E}_0)$	$H_3(\underline{E}_0)$	$H_4(\underline{E}_0)$	$H_5(\underline{E}_0)$	$H_6(\underline{E}_0)$
−1	$H_0(\underline{E}_{-1})$	$H_1(\underline{E}_{-1})$	$H_2(\underline{E}_{-1})$	$H_3(\underline{E}_{-1})$	$H_4(\underline{E}_{-1})$	$H_5(\underline{E}_{-1})$	$H_6(\underline{E}_{-1})$
−2	$H_0(\underline{E}_{-2})$	$H_1(\underline{E}_{-2})$	$H_2(\underline{E}_{-2})$	$H_3(\underline{E}_{-2})$	$H_4(\underline{E}_{-2})$	$H_5(\underline{E}_{-2})$	$H_6(\underline{E}_{-2})$
−3	$H_0(\underline{E}_{-3})$	$H_1(\underline{E}_{-3})$	$H_2(\underline{E}_{-3})$	$H_3(\underline{E}_{-3})$	$H_4(\underline{E}_{-3})$	$H_5(\underline{E}_{-3})$	$H_6(\underline{E}_{-3})$

Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$

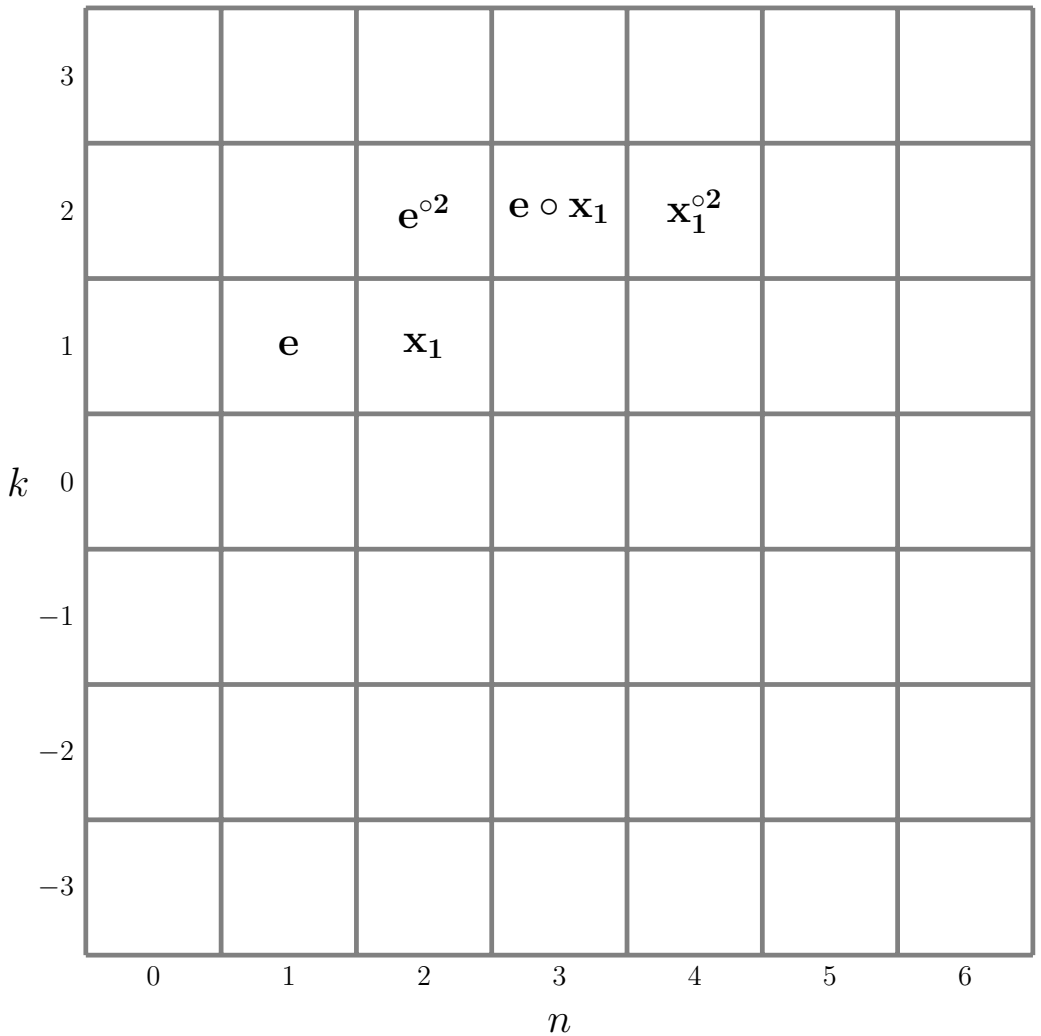
Star product ($\underline{E}_k$ is a loop space)



Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$

Circle product (E is a ring spectrum)



Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$

Circle product (E is a ring spectrum)

k \ n	0	1	2	3	4	5	6
3				$\mathbf{e}^{\circ 3}$	$\mathbf{e}^{\circ 2} \circ \mathbf{x}_1$	$\mathbf{e} \circ \mathbf{x}_1^{\circ 2}$	$\mathbf{x}_1^{\circ 3}$
2			$\mathbf{e}^{\circ 2}$	$\mathbf{e} \circ \mathbf{x}_1$	$\mathbf{x}_1^{\circ 2}$		
1		$\mathbf{e}$	$\mathbf{x}_1$				
0							
−1							
−2							
−3							

Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$

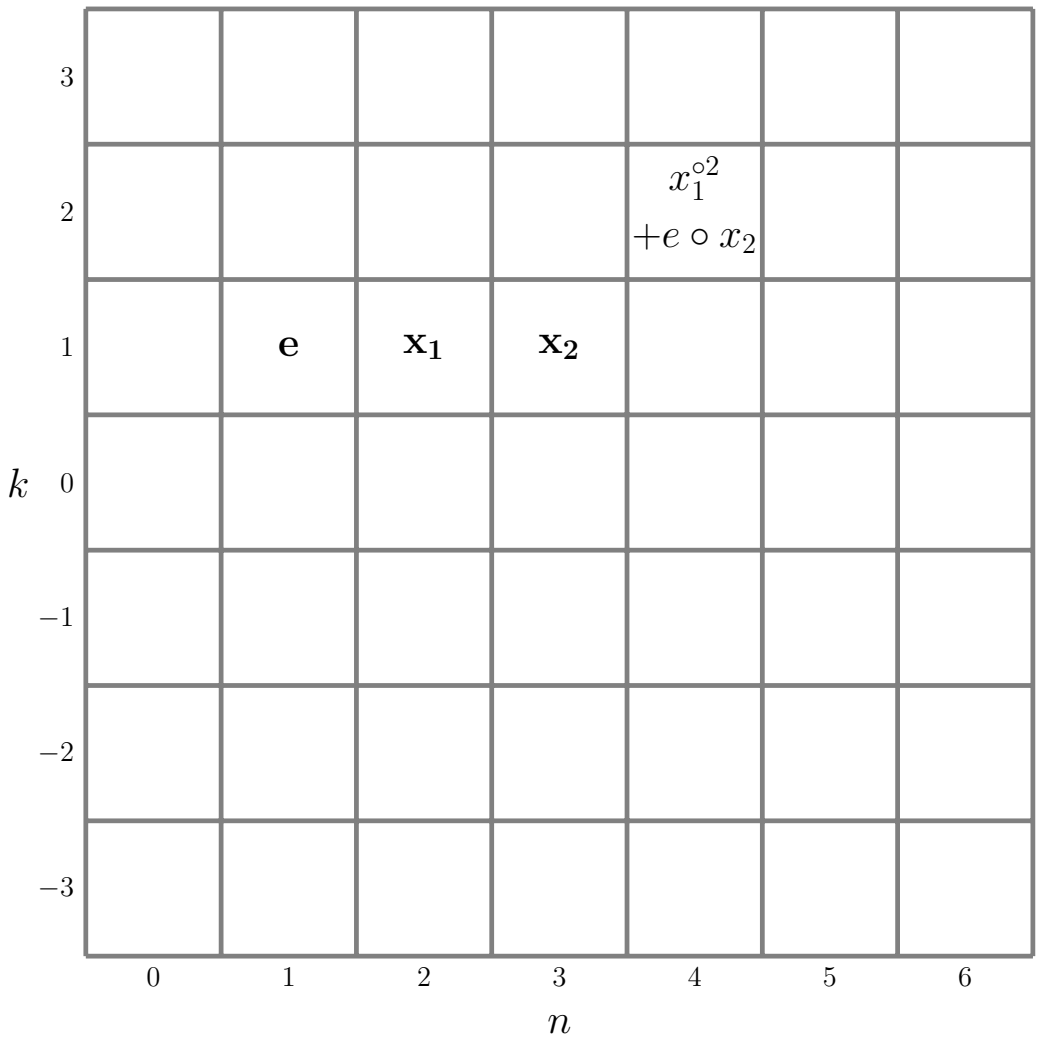
Circle product (E is a ring spectrum)

k \ n	0	1	2	3	4	5	6
3				$\mathbf{e}^{\circ 3}$	$\mathbf{e}^{\circ 2} \circ \mathbf{x}_1$	$\mathbf{e} \circ \mathbf{x}_1^{\circ 2}$	$\mathbf{x}_1^{\circ 3}$
2			$\mathbf{e}^{\circ 2}$	$\mathbf{e} \circ \mathbf{x}_1$	$\mathbf{x}_1^{\circ 2}$	$\mathbf{x}_1 \circ \mathbf{x}_2$	$\mathbf{x}_2^{\circ 2}$
1		$\mathbf{e}$	$\mathbf{x}_1$	$\mathbf{x}_2$			
0							
−1							
−2							
−3							

Hopf ring structure

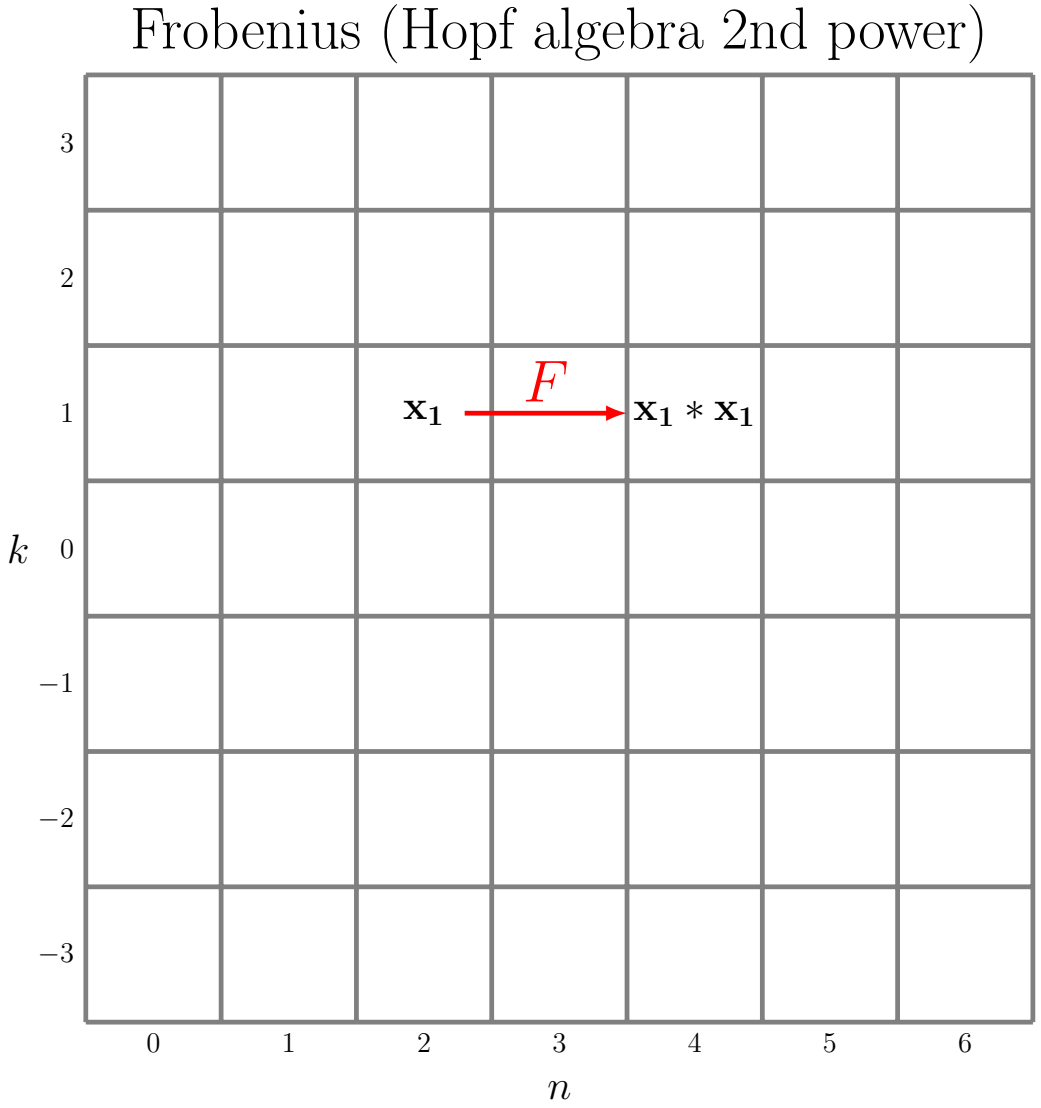
Picture of a Hopf ring $H_n(\underline{E}_k)$

Addition ($H_n(\underline{E}_k)$ is a group)



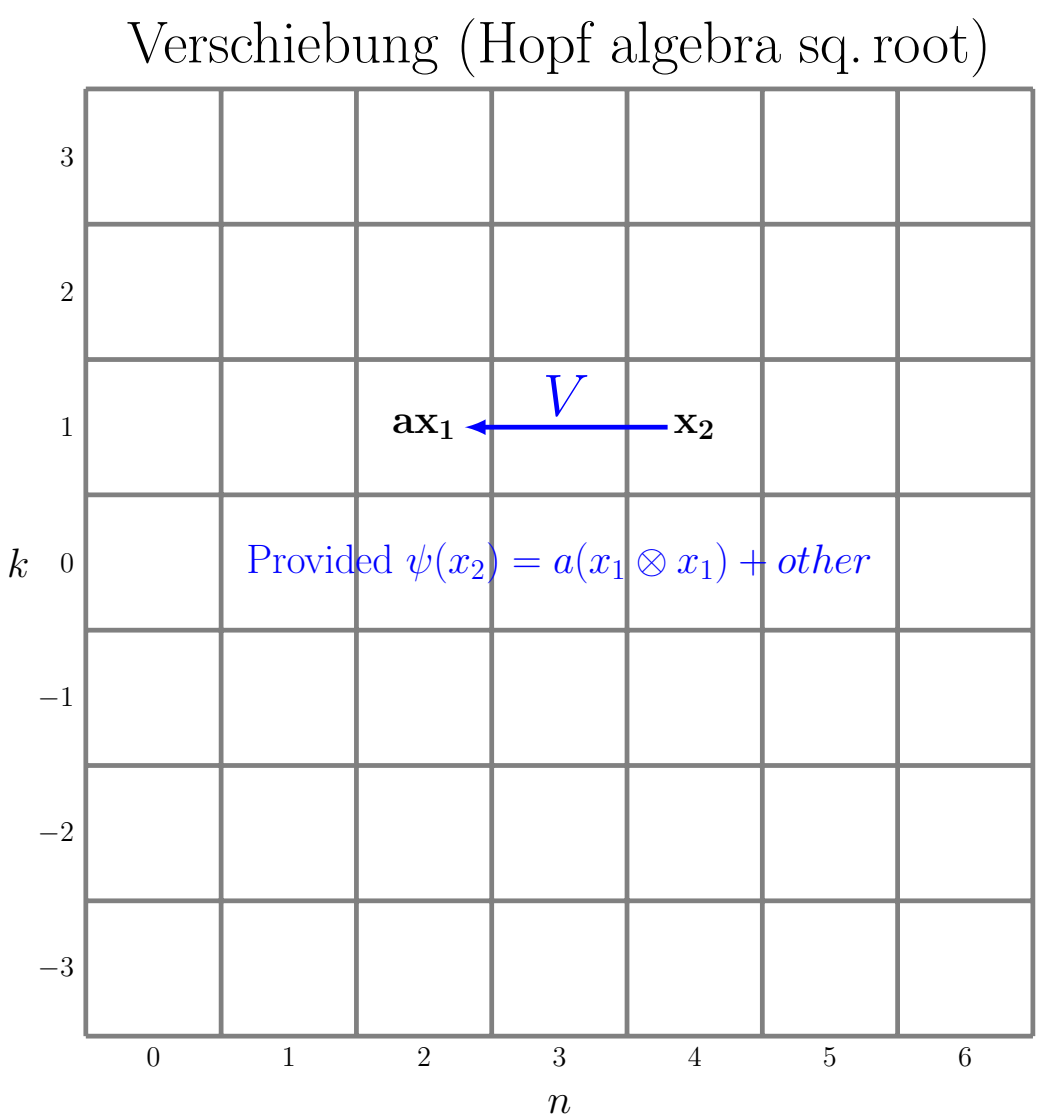
Hopf ring structure

Picture of a Hopf ring $H_n(\underline{E}_k)$



Hopf ring structure

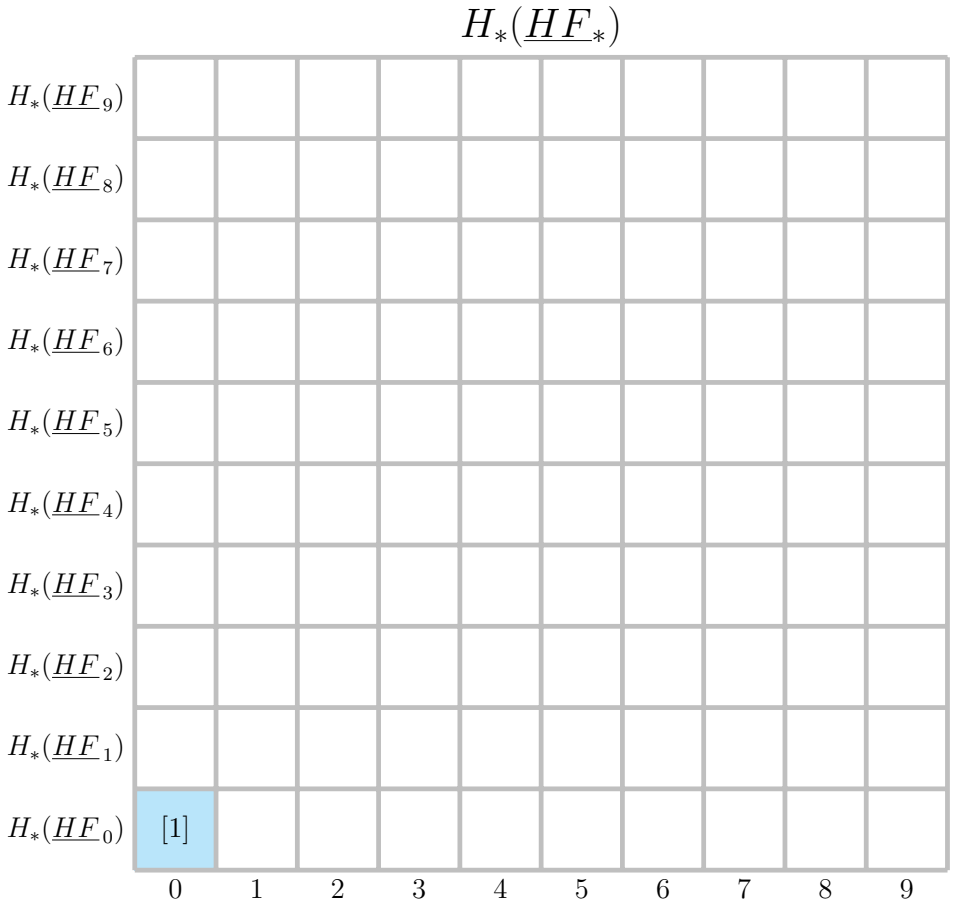
Picture of a Hopf ring $H_n(\underline{E}_k)$



Mod 2 Hopf ring for $H\mathbb{F}$

Example

$H_*(\underline{H\mathbb{F}}_0) = \mathbb{F}_2[\mathbb{F}_2]$



Mod 2 Hopf ring for $H\mathbb{F}$

Example

$H_*(\underline{H\mathbb{F}}_1) = H_*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{F}_2\{b_i \mid i \geq 0\}$ where $b_i * b_j = \binom{i+j}{i} b_{i+j}$.

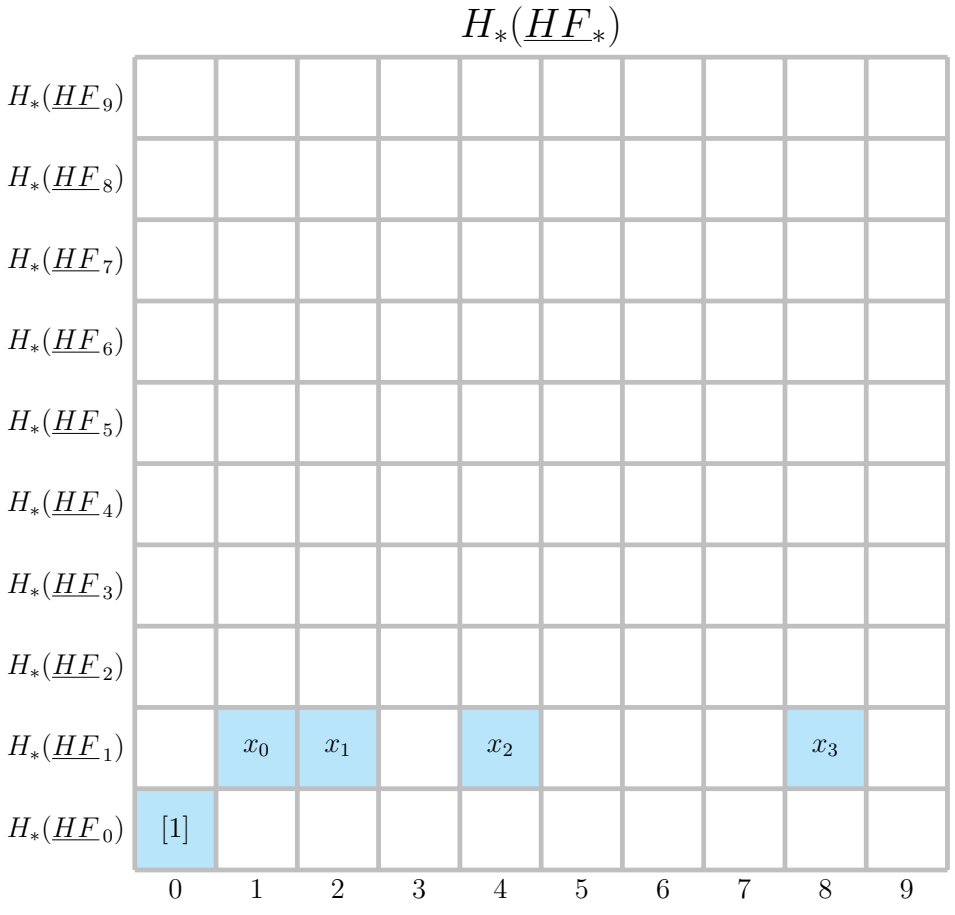
$H_*(\underline{HF}_*)$

$H_*(\underline{HF}_9)$										
$H_*(\underline{HF}_8)$										
$H_*(\underline{HF}_7)$										
$H_*(\underline{HF}_6)$										
$H_*(\underline{HF}_5)$										
$H_*(\underline{HF}_4)$										
$H_*(\underline{HF}_3)$										
$H_*(\underline{HF}_2)$										
$H_*(\underline{HF}_1)$	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
$H_*(\underline{HF}_0)$	[1]									
	0	1	2	3	4	5	6	7	8	9

Mod 2 Hopf ring for $H\mathbb{F}$

Example

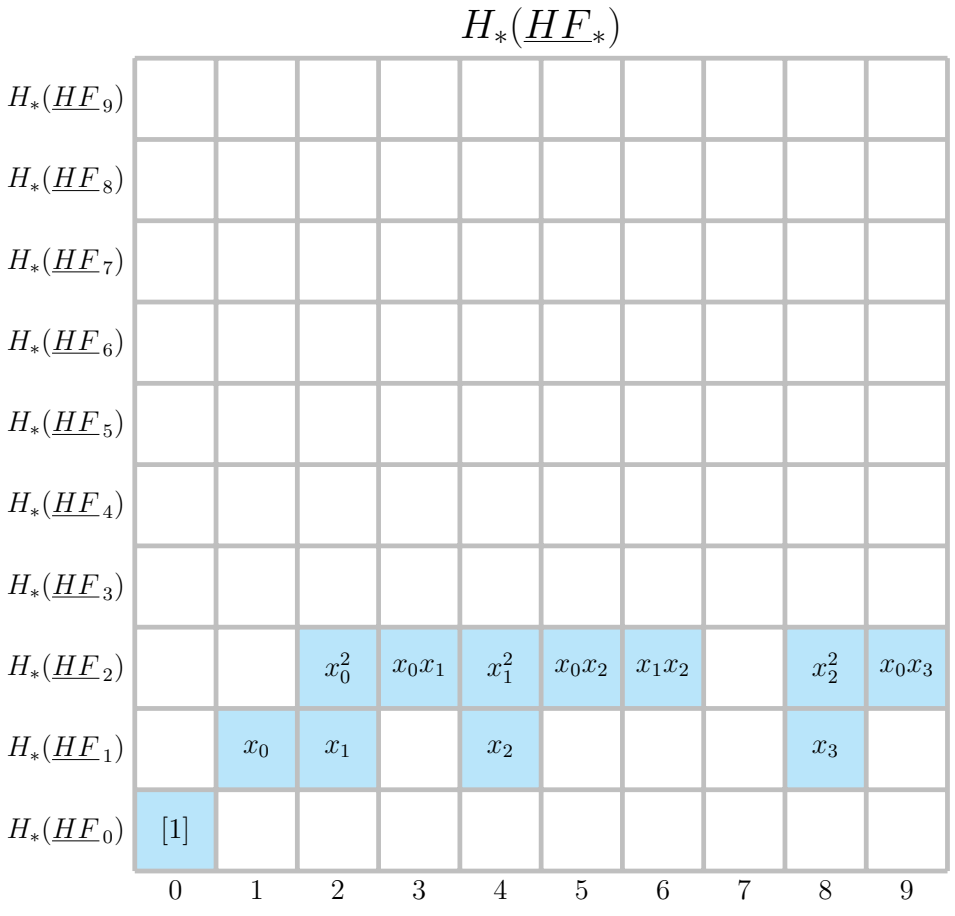
$*\text{-Ind}(H_*(\underline{H\mathbb{F}}_1)) = \mathbb{F}_2\{x_i \mid i \geq 0\}$ where $x_i = b_{2^i}$



Mod 2 Hopf ring for $H\mathbb{F}$

Example

$$H_*(\underline{H\mathbb{F}}_2) = E(x_{i_1} \circ x_{i_2} \mid 0 \leq i_1 \leq i_2)$$



Mod 2 Hopf ring for $H\mathbb{F}$

Example

$$H_*(\underline{H\mathbb{F}}_3) = E(x_{i_1} \circ x_{i_2} \circ x_{i_3} \mid 0 \leq i_1 \leq i_2 \leq i_3)$$

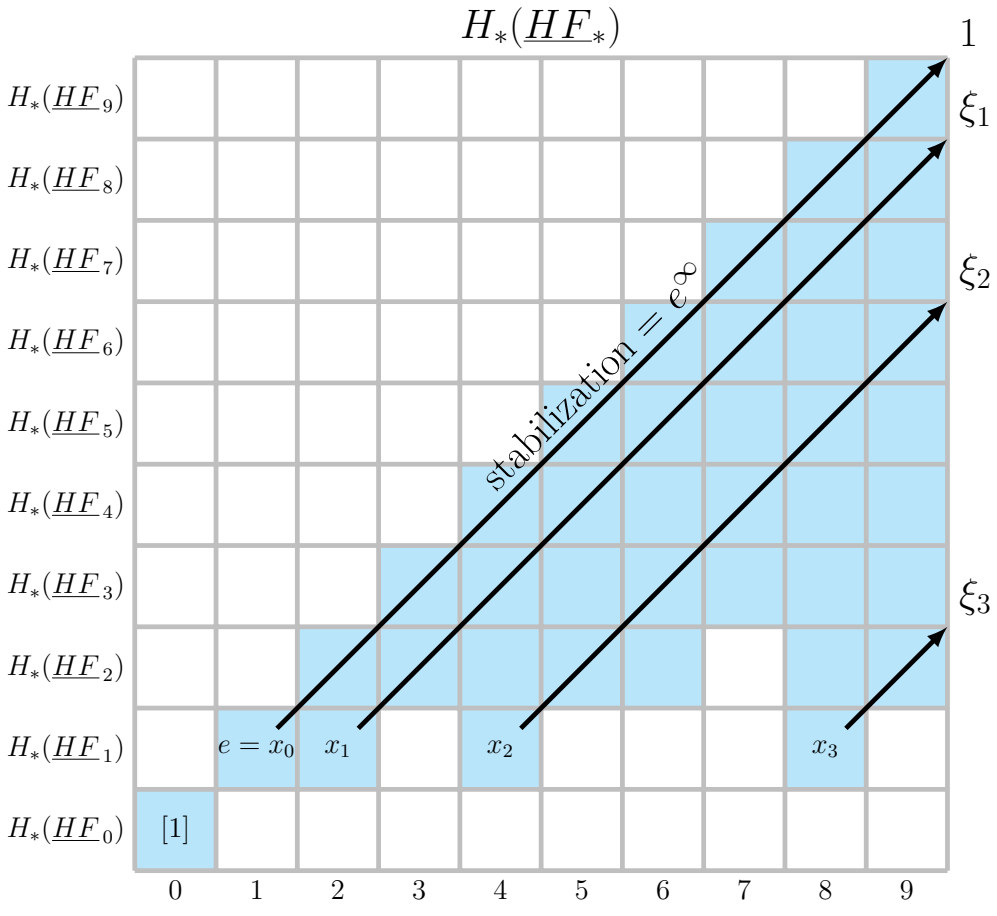
$H_*(\underline{HF}_*)$

	0	1	2	3	4	5	6	7	8	9
$H_*(\underline{HF}_9)$										
$H_*(\underline{HF}_8)$										
$H_*(\underline{HF}_7)$										
$H_*(\underline{HF}_6)$										
$H_*(\underline{HF}_5)$										
$H_*(\underline{HF}_4)$										
$H_*(\underline{HF}_3)$				x_0^3	$x_0^2x_1$	$x_0x_1^2$	$x_0^2x_2$ x_1^3	$x_0x_1x_2$	$x_1^2x_2$	$x_0x_2^2$
$H_*(\underline{HF}_2)$			x_0^2	x_0x_1	x_1^2	x_0x_2	x_1x_2		x_2^2	x_0x_3
$H_*(\underline{HF}_1)$		x_0	x_1		x_2				x_3	
$H_*(\underline{HF}_0)$	[1]									

Mod 2 Hopf ring for $H\mathbb{F}$

Example

$$\lim_{n\to\infty} H_*(\underline{H\mathbb{F}}_n) = H_*(H\mathbb{F}) = A^\vee = \mathbb{F}_2[\xi_i \mid i \geq 1]$$



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Adams spectral sequence for ko
Dieudonné ring for ko
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References

Dieudonné modules over $\mathbb{Z}_2^\wedge$

Definition

A Dieudonné module over $\mathbb{Z}_2^\wedge$ is a graded abelian group $M_{*,k} = \{M_{n,k}\}_{n\geq 0}$ with maps $F : M_{n,k} \to M_{2n,k}$ and $V : M_{2n,k} \to M_{n,k}$ such that

$$\begin{cases} FV = VF = 2 \\ V = id \text{ on } M_0 \\ 2^{i+1}M_{2^i m} = 0 \text{ if } m \text{ is odd.} \end{cases}$$

Dieudonné rings over $\mathbb{Z}_2^\wedge$

Definition

A Dieudonné ring is a family $\{M_{*,k}\}_{k\in\mathbb{Z}}$ of Dieudonné modules with a product

$$\circ : M_{m,j} \underset{\mathbb{Z}_2^\wedge}{\otimes} M_{n,k} \;\to\; M_{m+n,j+k}$$

satisfying the Frobenius reciprocity relations

$$\begin{aligned} V(x\circ y) &= V(x)\circ V(y),\\ F(x)\circ y &= F(x\circ V(y)),\\ x\circ F(y) &= F(V(x)\circ y). \end{aligned}$$

Hopf and Dieudonné rings



Theorem (Schoeller, Ravenel, Goerss)

(Schoeller, Ravenel) The category H *of Hopf algebras over* $\mathbb{F}_2$ *is equivalent to the category* D *of Dieudonné modules over* $\mathbb{Z}_2^\wedge$.

(Goerss) There exist bilinear products $\underset{\mathrm{H}}{\boxtimes}$ *and* $\underset{\mathrm{D}}{\boxtimes}$, *and the Dieudonné functor* $D_*(-)$ *is symmetric monoidal. Thus,*

$$D_*(-) : \left\{ \begin{array}{c} \textit{coalgebraic rings} \\ \textit{over } \mathbb{F}_2 \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Dieudonné rings} \\ \textit{over } \mathbb{Z}_2^\wedge \end{array} \right\}$$

is an equivalence of categories between the category HR *of Hopf rings over* $\mathbb{F}_2$ *and the category* DR *of Dieudonné rings over* $\mathbb{Z}_2^\wedge$.

Remark: The Dieudonné functor is represented by a Hopf algebras for which the ~~Witt~~ polynomials are primitive.

B.G. spectra and Dieudonné rings



Theorem (Goerss)

For any ring spectum E and all $(n,k) \in \mathbb{N} \times \mathbb{Z}$, there is a surjective homomorphism of Dieudonné rings

$$T : E_{n-k}(B(n)) \to D_n(H.(\underline{E}_k))$$

that respects the Frobenius and Verschiebung, and is an isomorphism when n is even.

Calculation method: Use the Adams spectral sequence to calculate $E_*(B(*))$:

$$E_2^{*,*}(A, E \wedge B(*)) \implies E_*(B(*)) \xrightarrow{T} D_*(H.(\underline{E}_*)) \xleftarrow{\equiv} H_*(\underline{E}_*).$$

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Mod 2 Steenrod algebra

HOPE COLLEGE

Steenrod algebra

- $A = H^*(\mathrm{H}\mathbb{F})$ generated by Sq^{2^i}.
- $A^\vee = H_*(\mathrm{H}\mathbb{F}) = \mathbb{F}_2[\xi_i \mid i \geq 1]$ where $deg(\xi_i) = 2^i - 1$. Let $\zeta_i = \chi(\xi_i)$.

Connective real ko-theory

- $H^*(\mathrm{ko}) = A//A(1) = A \otimes_{A(1)} \mathbb{F}_2$
- $H_*(\mathrm{ko}) = \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \ldots]$
- Note: $A(1) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$

Connective tmf-theory

- $H^*(\mathrm{tmf}) = A//A(2) = A \otimes_{A(2)} \mathbb{F}_2$
- $H_*(\mathrm{tmf}) = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots]$
- Note: $A(2) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4 \rangle$

Steenrod action

Let

$$\mathrm{Sq} = \sum_{i \geq 0} \mathrm{Sq}^i.$$

The right action of the Steenrod algebra A on its dual $A^\vee$ is

$$\xi_n \cdot \mathrm{Sq} = \xi_n - \xi_{n-1},$$

$$\zeta_n \cdot \mathrm{Sq} = \sum_{i=0}^{n} \zeta_{n-i}^{2^i}.$$

Example

$\xi_2 \cdot \mathrm{Sq} = \xi_2 + \xi_1$ means $\xi_2 \cdot \mathrm{Sq}^0 = \xi_2$ and $\xi_2 \cdot \mathrm{Sq}^2 = \xi_1$ and all other Sq^i are zero.

Note: these formulas carry over to x_n and z_n by replacing 1 with $x_0 = z_0$.

Brown-Gitler Modules



Theorem (Brown & Gitler)

For each $n \in \mathbb{N}$ there is a 2-complete spectrum $B(2n) \simeq B(2n+1)$ satisfying

- $H^*(B(2n)) = A/A\{\chi(\mathrm{Sq}^i) \mid i > n\}$ *as left A-modules, and*
- *If $\iota : B(2n) \to \mathrm{H}\mathbb{F}_2$ classifies $1 \in H^0(B(2n))$, then $\iota_* : B(2n)_k(X) \to H_k(X)$ is an epimorphism for all complexes X and $0 \leq k \leq 2n+1$.*

Theorem (Brown & Gitler)

$H^*(B(2n))$ has a basis $\chi(\mathrm{Sq}^I)$ over $\mathbb{F}_2$ where $\mathrm{Sq}^I = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_\ell}$ is admissible and $i_1 \leq n$.

Weight function

HOPE COLLEGE

Definition

Define the ζ weight function $wt_\zeta : A^\vee \to \mathbb{N}$ on monomials by $wt_\zeta(1) = 0$ and $wt_\zeta(\zeta_i) = 2^{i-1}$, on products by $wt_\zeta(xy) = wt_\zeta(x) + wt_\zeta(y)$, and on sums by $wt_\zeta(\sum \zeta^I) = \max\{wt_\zeta(\zeta^I)\}$. If $\sum \zeta^I = \sum \xi^K$, set $wt_\zeta(\sum \xi^K) = \max\{wt_\zeta(\zeta^I)\}$.

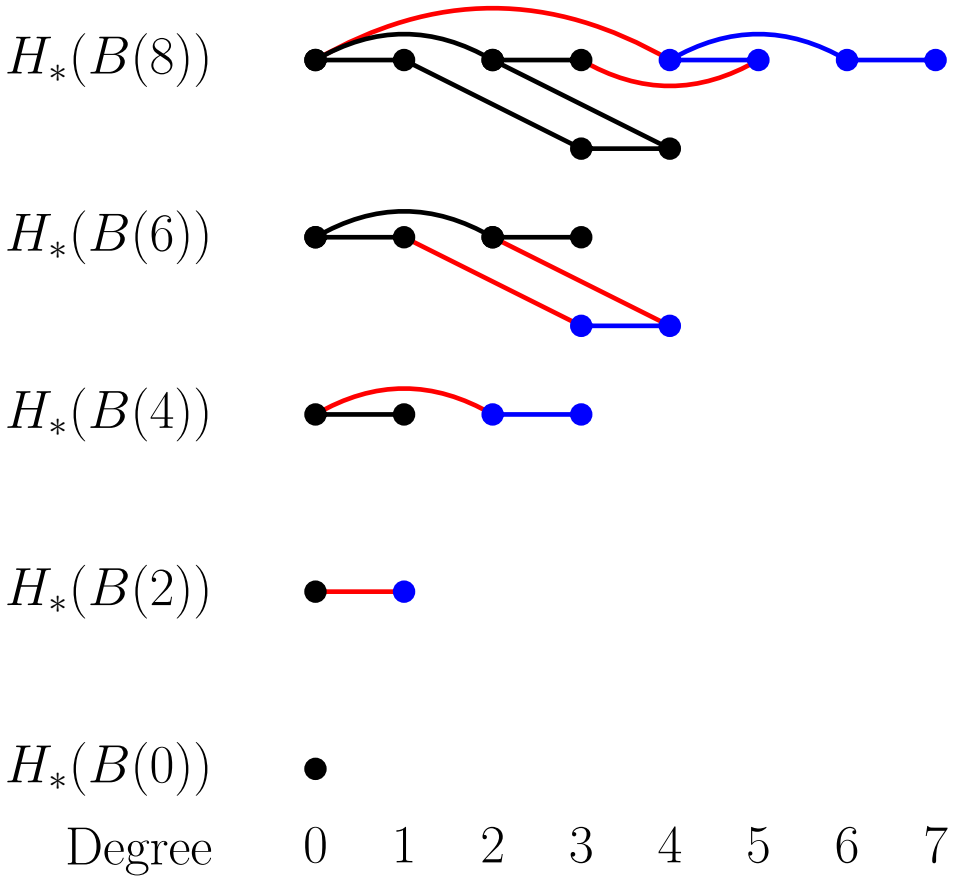
Theorem (Brown & Gitler)

$H_*(B(2n)) = \mathbb{F}_2\{\zeta^I \in A^\vee \mid \mathrm{wt}_\zeta(\zeta^I) \leq n\}$ *as right A modules.*

Brown-Gitler Homology

HOPE COLLEGE

Example



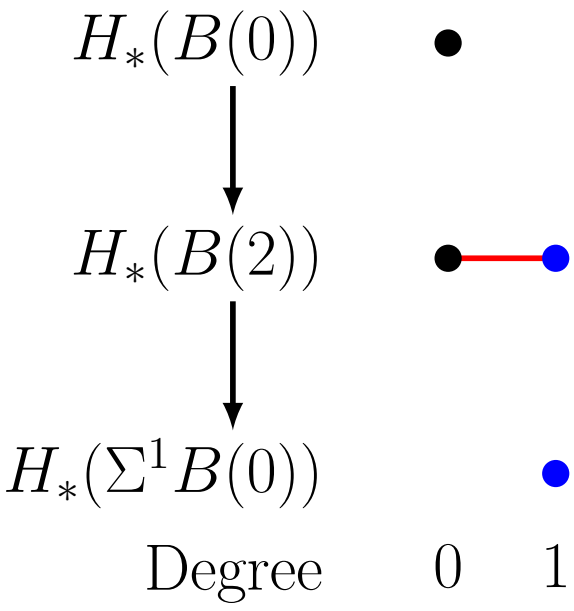
Brown-Gitler Cofiber Sequence

Theorem (Brown & Gitler)

There is a cofiber sequence $B(2n-2) \xrightarrow{i} B(2n) \xrightarrow{v} \Sigma^n B(n)$ *for all* $n \geq 1$ *that induces a short exact sequence of right A modules*

$$0 \to H_*(B(2n-2)) \xrightarrow{i_*} H_*(B(2n)) \xrightarrow{v_*} \Sigma^n H_*(B(n)) \to 0.$$

Example



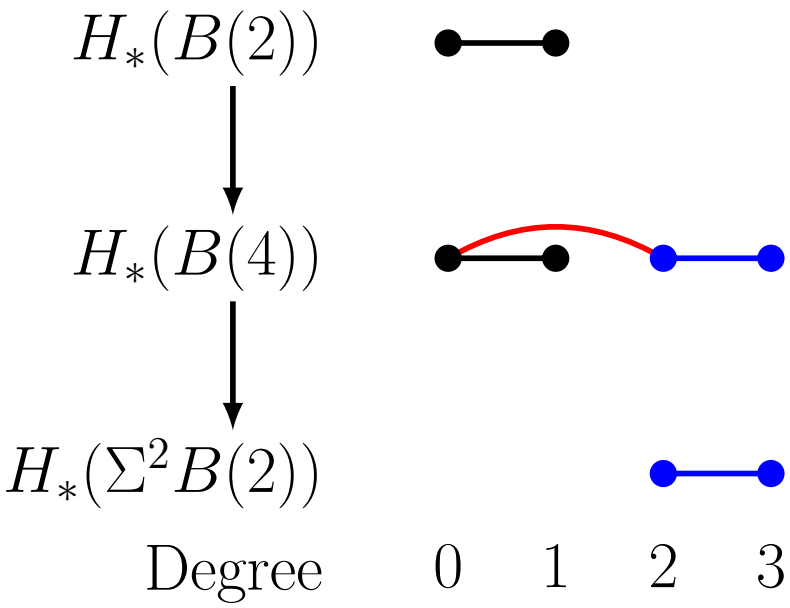
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Example



Brown-Gitler Cofiber Sequence

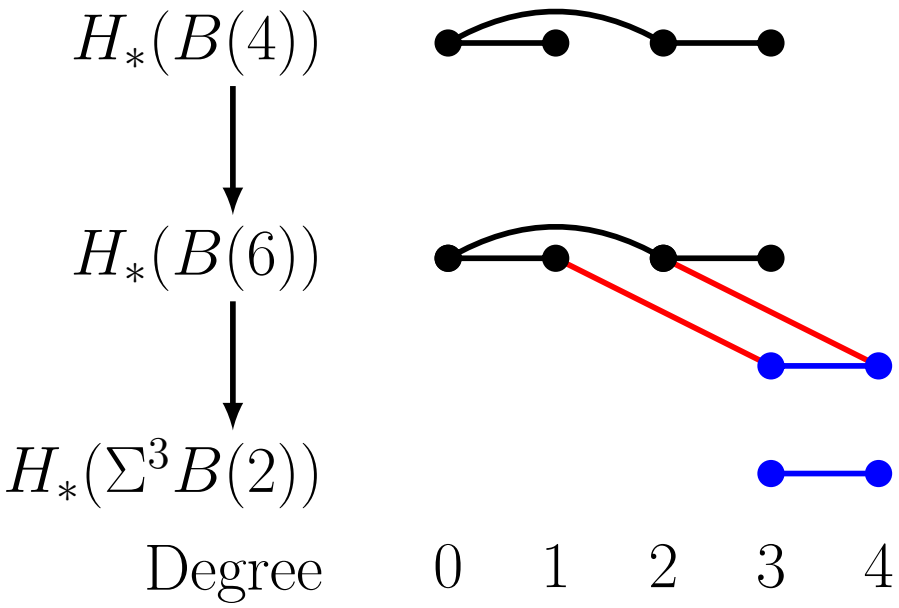


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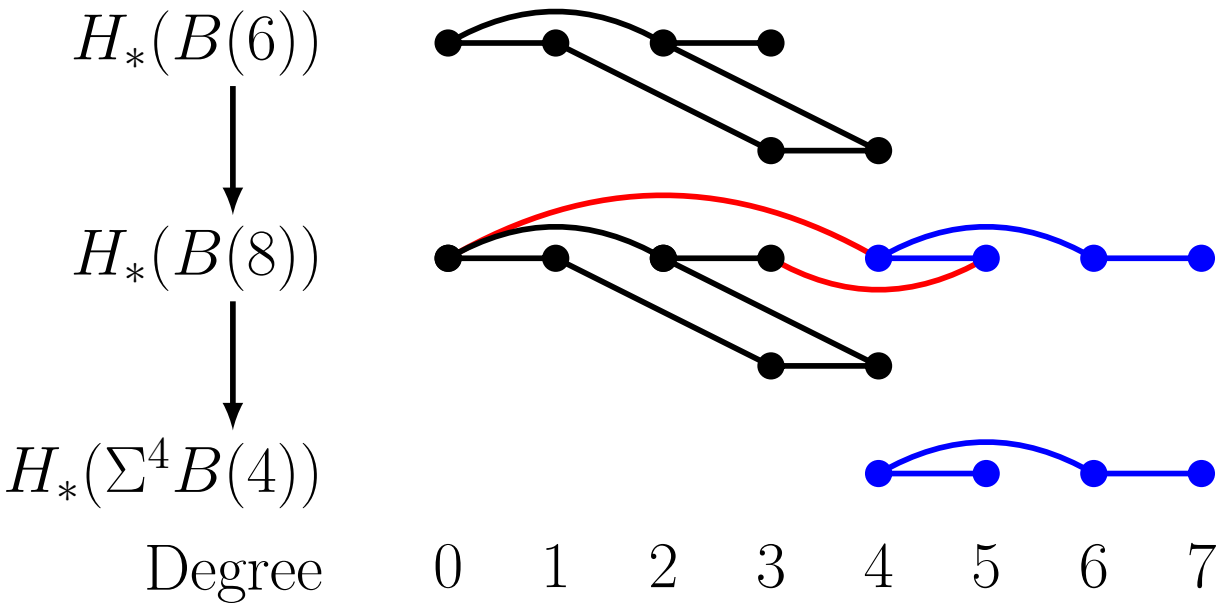
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Theorem (Brown & Gitler)

There is a cofiber sequence $B(2n-2) \xrightarrow{i} B(2n) \xrightarrow{v} \Sigma^n B(n)$ *for all* $n \geq 1$ *that induces a short exact sequence of right A modules*

$$0 \to H_*(B(2n-2)) \xrightarrow{i_*} H_*(B(2n)) \xrightarrow{v_*} \Sigma^n H_*(B(n)) \to 0.$$

Example



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Lambda algebra



Definition (Bousfield, Curtis, Kan, Quillen, Rector, Schlesinger)

The Lambda algebra Λ is the associative bigraded differential algebra over $\mathbb{F}_2$ with generators λ_a, $a \geq -1$, of bidegree $(1, a+1) = (s, t)$ modulo the two-sided ideal generated by the relations

$$\lambda_a \lambda_b = \sum_{\lceil (b-2a)/2 \rceil \leq c < b-2a} \binom{c-1}{2c-b+2a} \lambda_{a+c} \lambda_{b-c}, \text{ if } 0 \leq 2a < b$$

and the left ideal $\Lambda\{\lambda_{-1}\}$. Its differential $d_1(\lambda_b) = \lambda_{-1}\lambda_b$ is a derivation.

Adams S.S.



Theorem (Bousfield, Curtis, Kan, Quillen, Rector, Schlesinger, Adams)

Let X be a complex or spectrum of finite type, and let E be a spectrum. The E_1 term of the Adams spectral sequence for E homology is the differential right (Λ, d_1) module

$$E_1^{*,*}(A, E \wedge X) = H_*(E \wedge X) \otimes \Lambda$$

with differential $d_1(z \otimes \lambda_I) = \sum_{i \geq 0} z \cdot \mathrm{Sq}^i \otimes \lambda_{i-1}\lambda_I$. *Its E_2 term is*

$$E_2^{s,t}(A, E \wedge X) = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, H_*(E \wedge X)) \implies E_{t-s}(X).$$

Change of rings



Theorem (Adams)

If E is a ring spectrum such that $H^*(E) = A \otimes_C \mathbb{F}_2$ for some sub Hopf algebra $C \subset A$, then

$$\mathrm{Ext}_C^{s,t}(\mathbb{F}_2, H_*(X)) \cong \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, H_*(E \wedge X)).$$

- When $E = \mathrm{ko}$, use $C = A(1)$.
- When $E = \mathrm{tmf}$, use $C = A(2)$.

Zero line of Adams S.S.

Definition (Pearson)

The destabilization function

$$\epsilon^{-\infty} : H_*(\mathrm{H}\mathbb{F}) \to H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(*)) \otimes \Lambda$$

is given by $\epsilon^{-\infty}(y) = \tau(\chi \otimes 1(\psi(y))) \otimes 1$, where $\tau(x \otimes y) = y \otimes x$.

Motivation: Take advantage of Milnor's equation

$$\mu(\chi \otimes 1(\psi(\xi_i))) = \mu\left(\sum_{i=0}^{n} \zeta_{n-i}^{2^i} \otimes \xi_i\right) = \sum_{i=0}^{n} \zeta_{n-i}^{2^i} \xi_i = 0.$$

Theorem (Pearson)

The destabilization function is a ring homomorphism and every element in the image of the destabilization is a permanent cycle in the Adams spectral sequence.

Zero line of Adams S.S.



Example

On generators of $H_*(\mathrm{H}\mathbb{F})$, the destabilization is

$$\epsilon^{-\infty}(\xi_n) = \sum_{i=0}^{n} \xi_i \otimes \zeta_{n-i}^{2^i} \otimes 1 \in H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(2^n)) \otimes \Lambda,$$

$$\epsilon^{-\infty}(\zeta_n) = \sum_{i=0}^{n} \zeta_{n-i}^{2^i} \otimes \xi_i \otimes 1 \in H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(2^{n+1}-2)) \otimes \Lambda.$$

Theorem (Pearson)

For $E = \mathrm{ko}, \mathrm{tmf}, k(r)$, *etc., there is a bijection*

$$\epsilon^{-\infty} : \left\{ \sum \zeta^I \in H_*(E) \mid wt_\xi(\sum \zeta^I) \leq n \right\} \to E_2^{0,t}(A, E \wedge B(2n))$$

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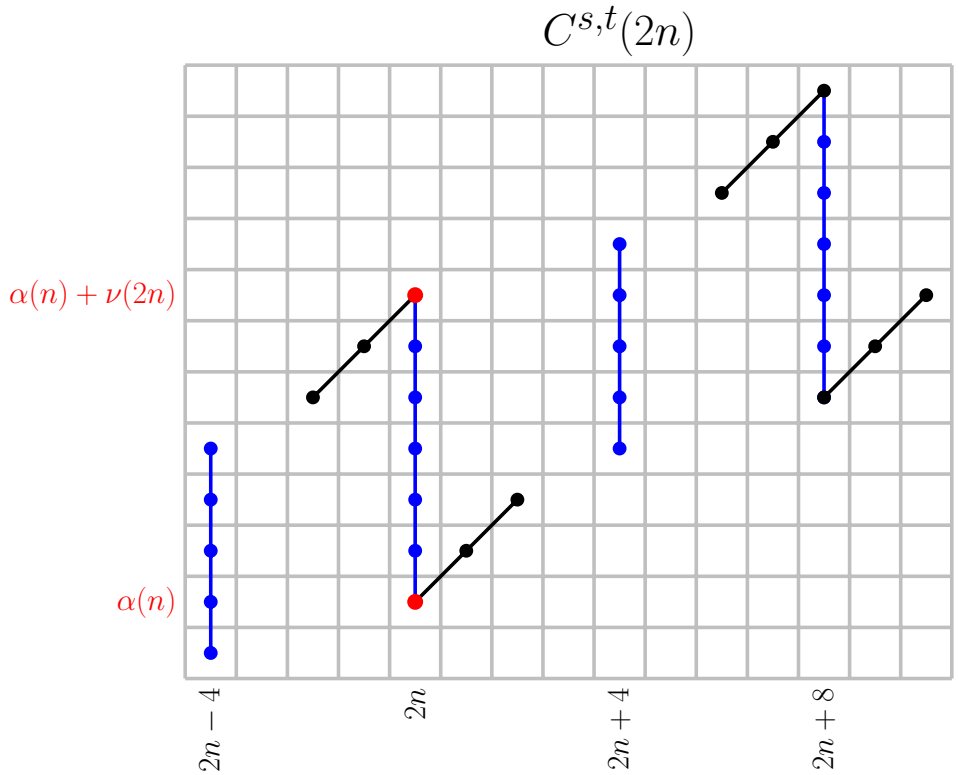
References

Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

HOPE COLLEGE

Theorem (Pearson)

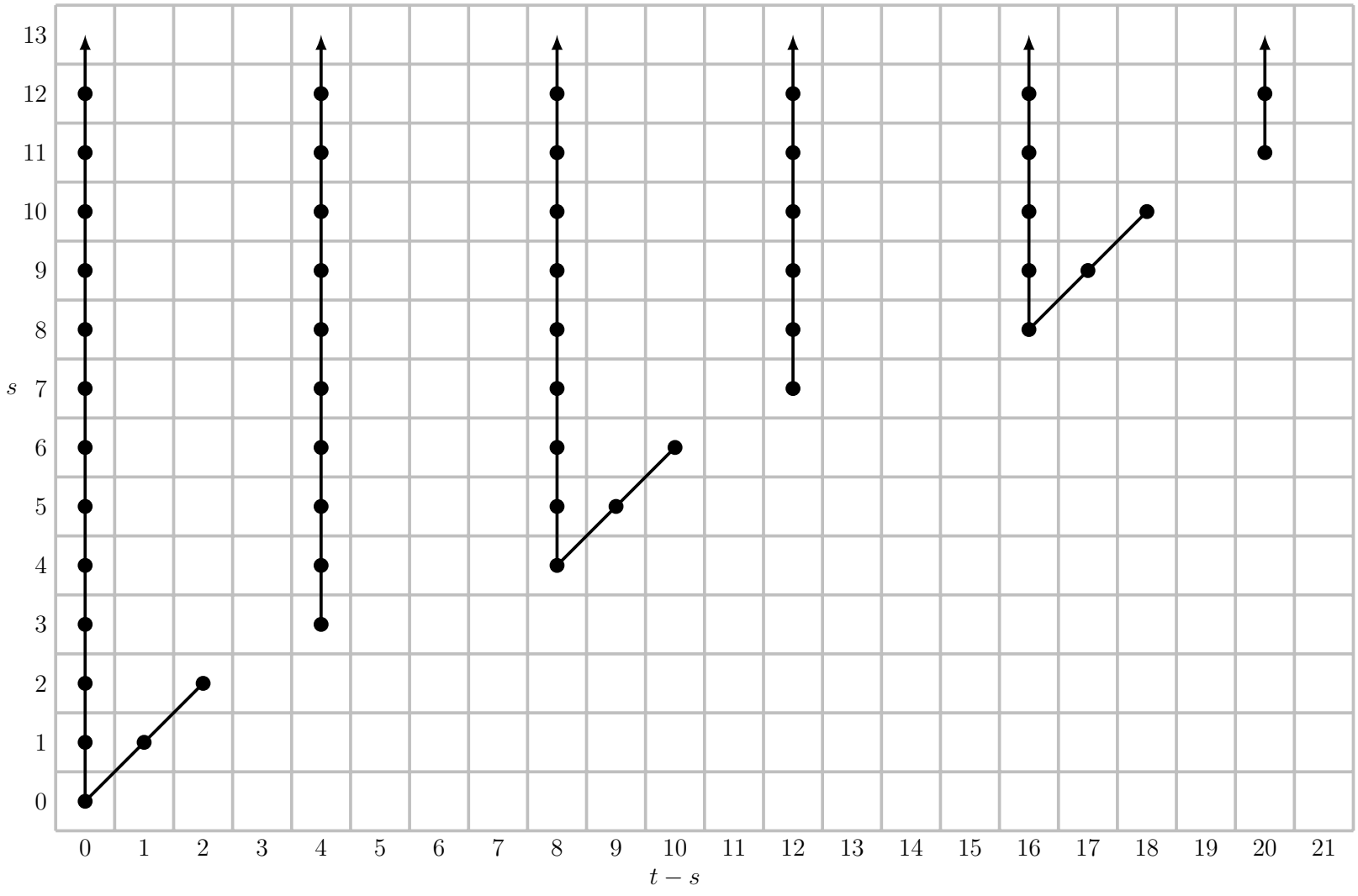
For $n \geq 1$, there is a map $C^{s,t}(2n) \to \mathrm{Ext}^{s,t}_{A(1)}(\mathbb{F}_2, H_*(B(2n)))$ which is injective for $s = 0$ and an isomorphism for $s \geq 1$. The spectral sequence for $\mathrm{ko}_*(B(2n))$ collapses.



Note: $\alpha(n)$ is the number of ones in the 2-adic expansion of n, $\nu(n)$ is the number of times 2 divides n, and the vertical towers vanish when $\nu(n) = 0$.

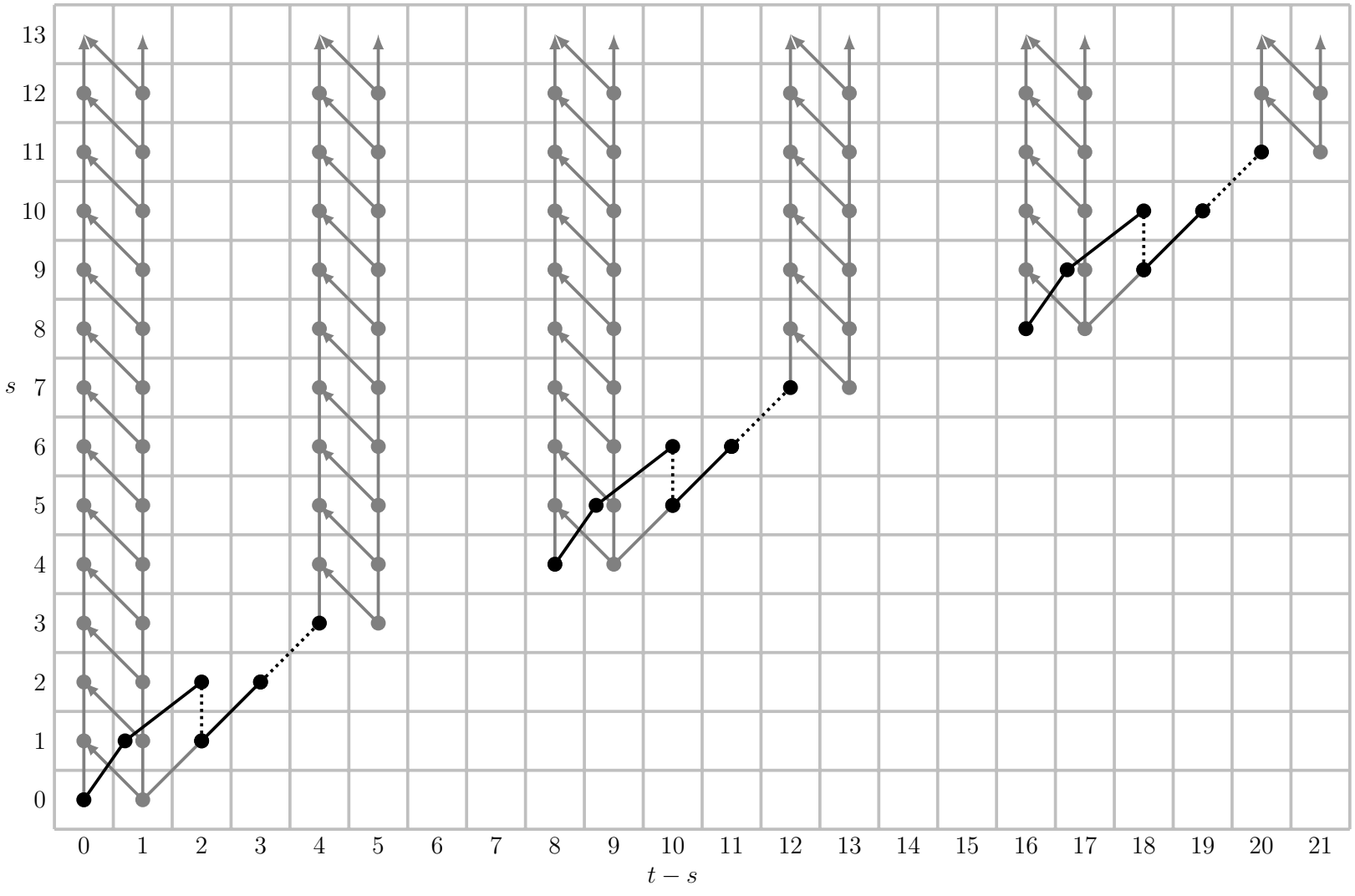
Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(0))) \Rightarrow \mathrm{ko}_{t-s}(B(0))$$



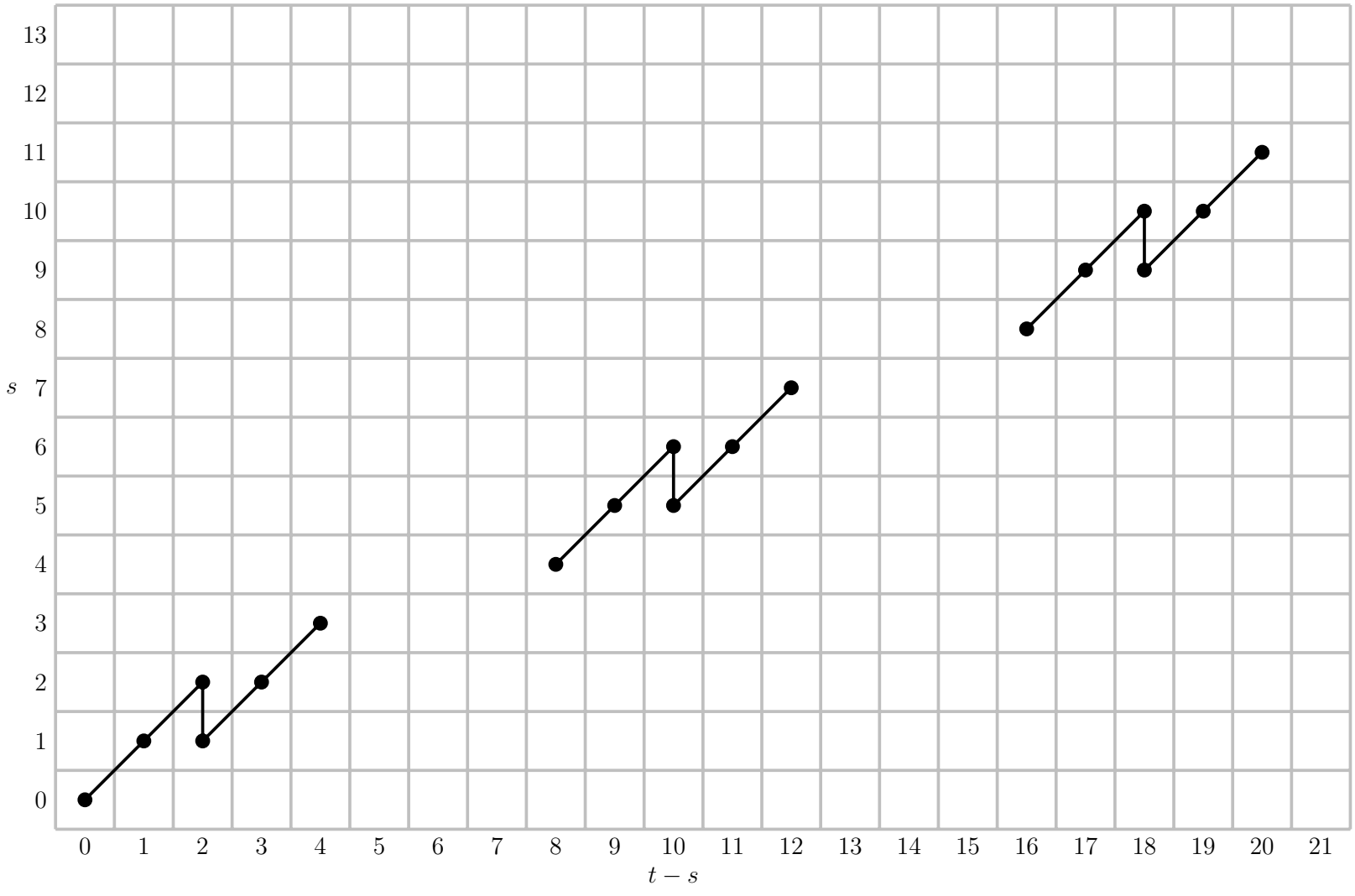
Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

Long exact sequence in $Ext_{A(1)}$ from $B(0) \to B(2) \to \Sigma^1 B(0)$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

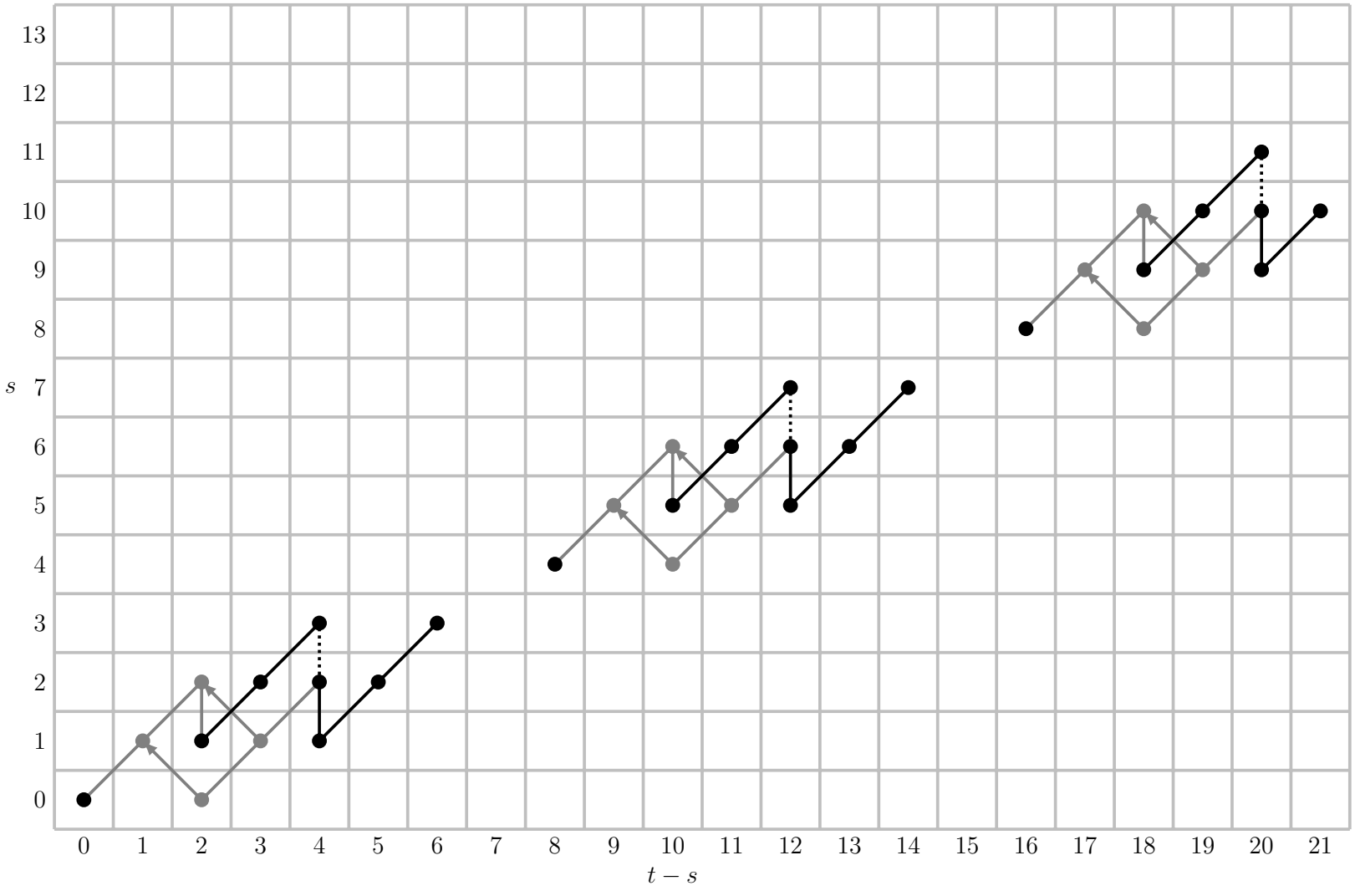
$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(2))) \Rightarrow \mathrm{ko}_{t-s}(B(2))$$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

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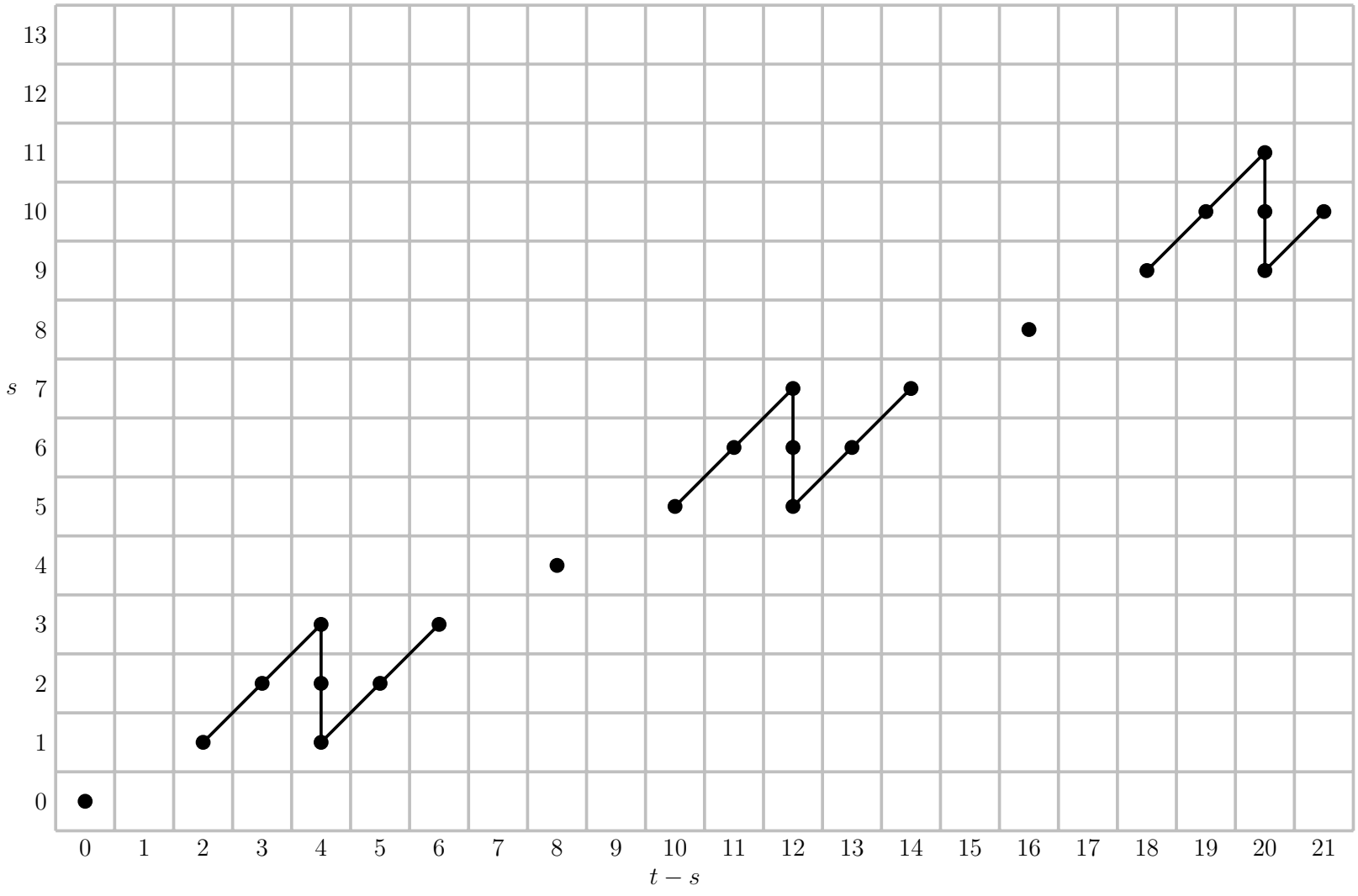
Long exact sequence in $Ext_{A(1)}$ from $B(2) \to B(4) \to \Sigma^2 B(2)$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

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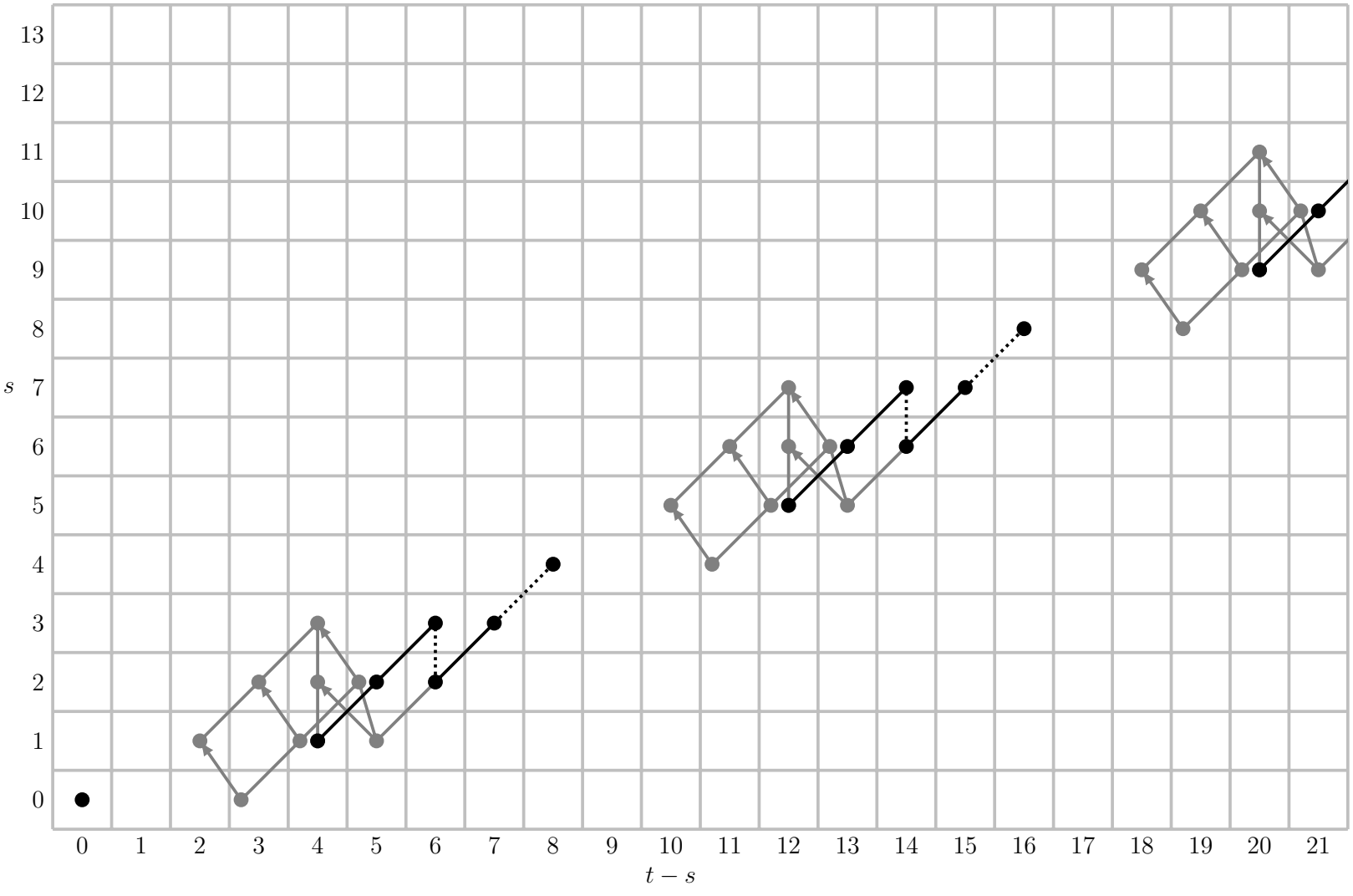
$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(4))) \Rightarrow \mathrm{ko}_{t-s}(B(4))$$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

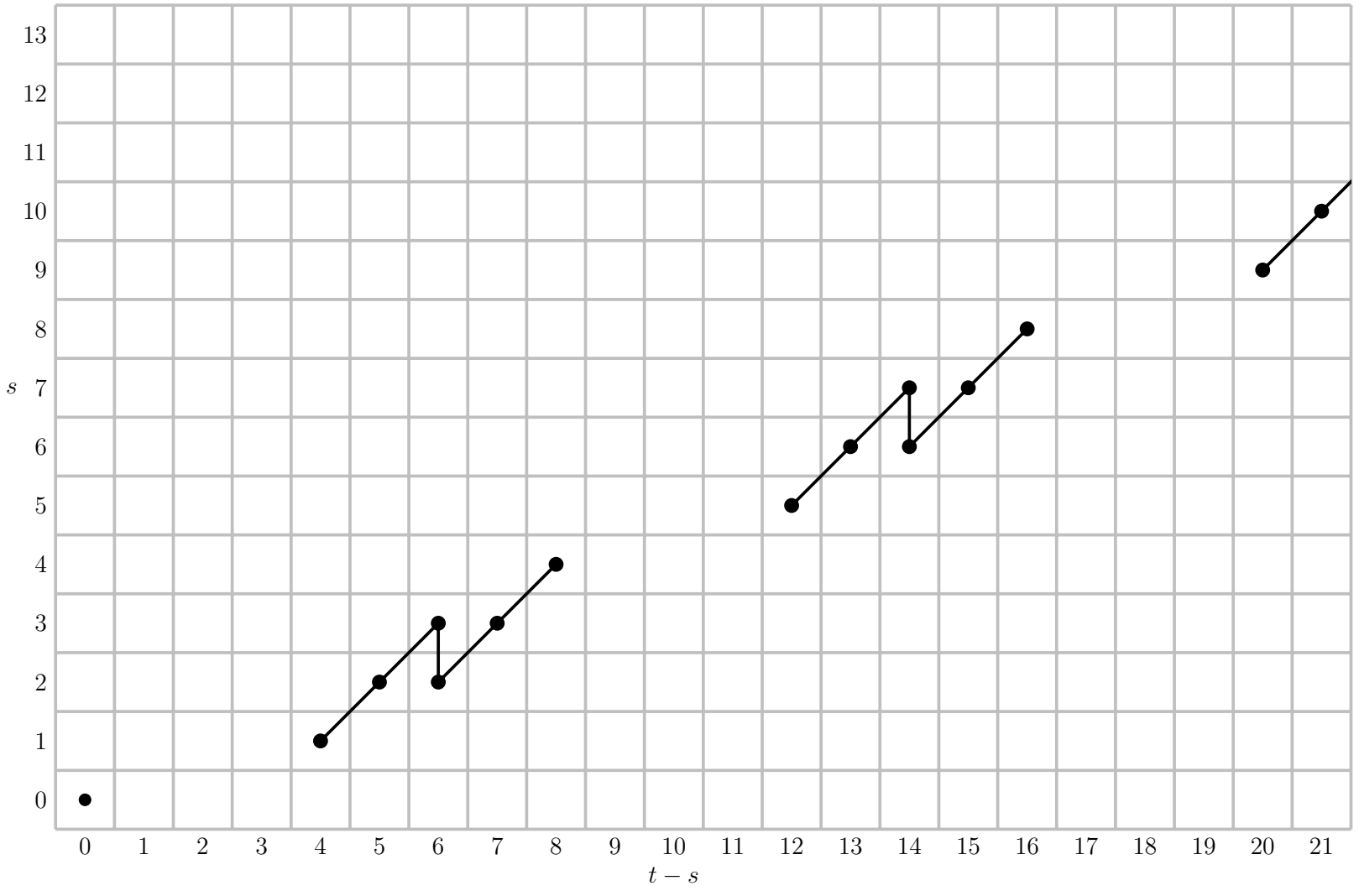
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Long exact sequence in $Ext_{A(1)}$ from $B(4) \to B(6) \to \Sigma^3 B(2)$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

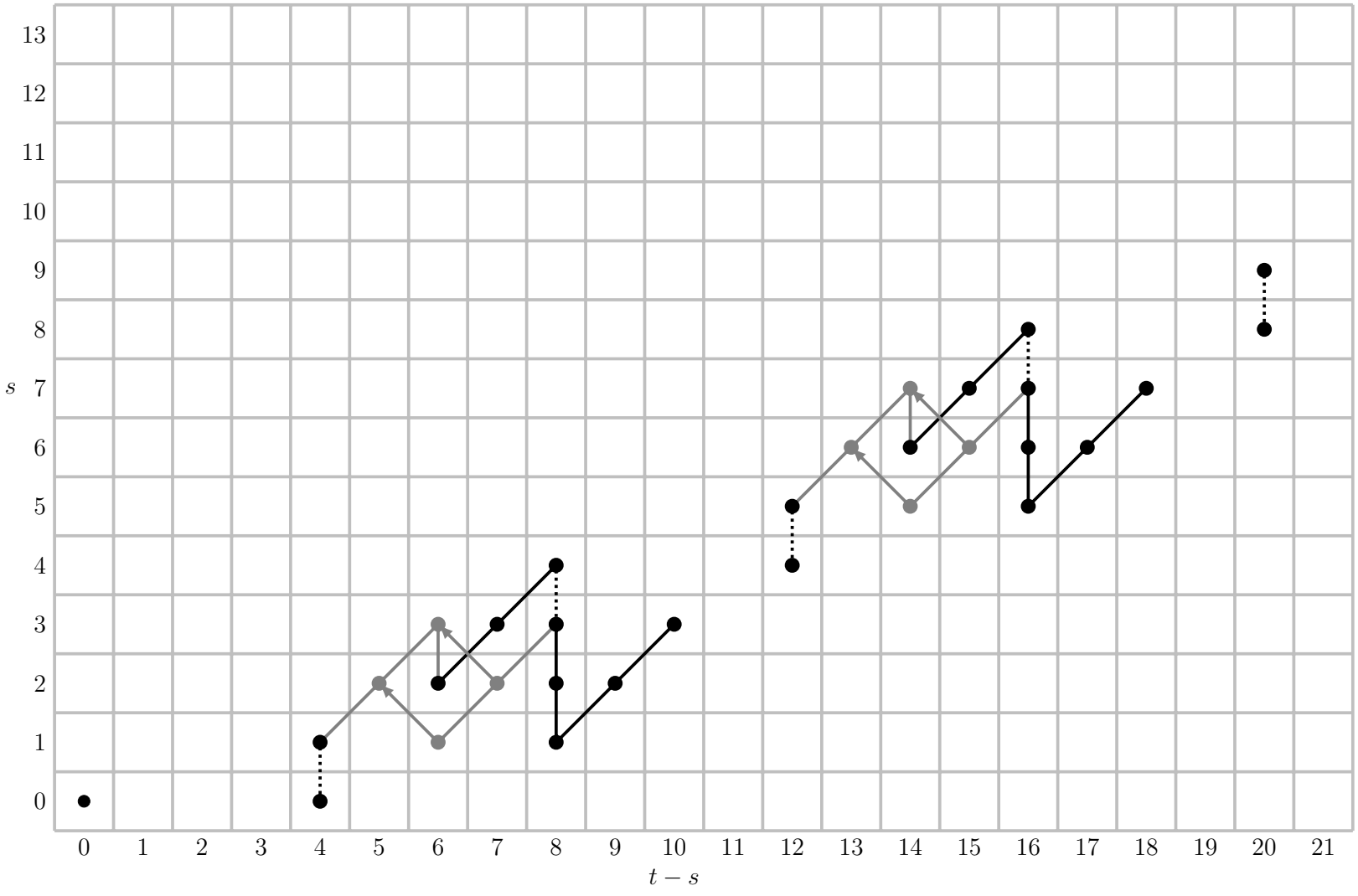
$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(6))) \Rightarrow \mathrm{ko}_{t-s}(B(6))$$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

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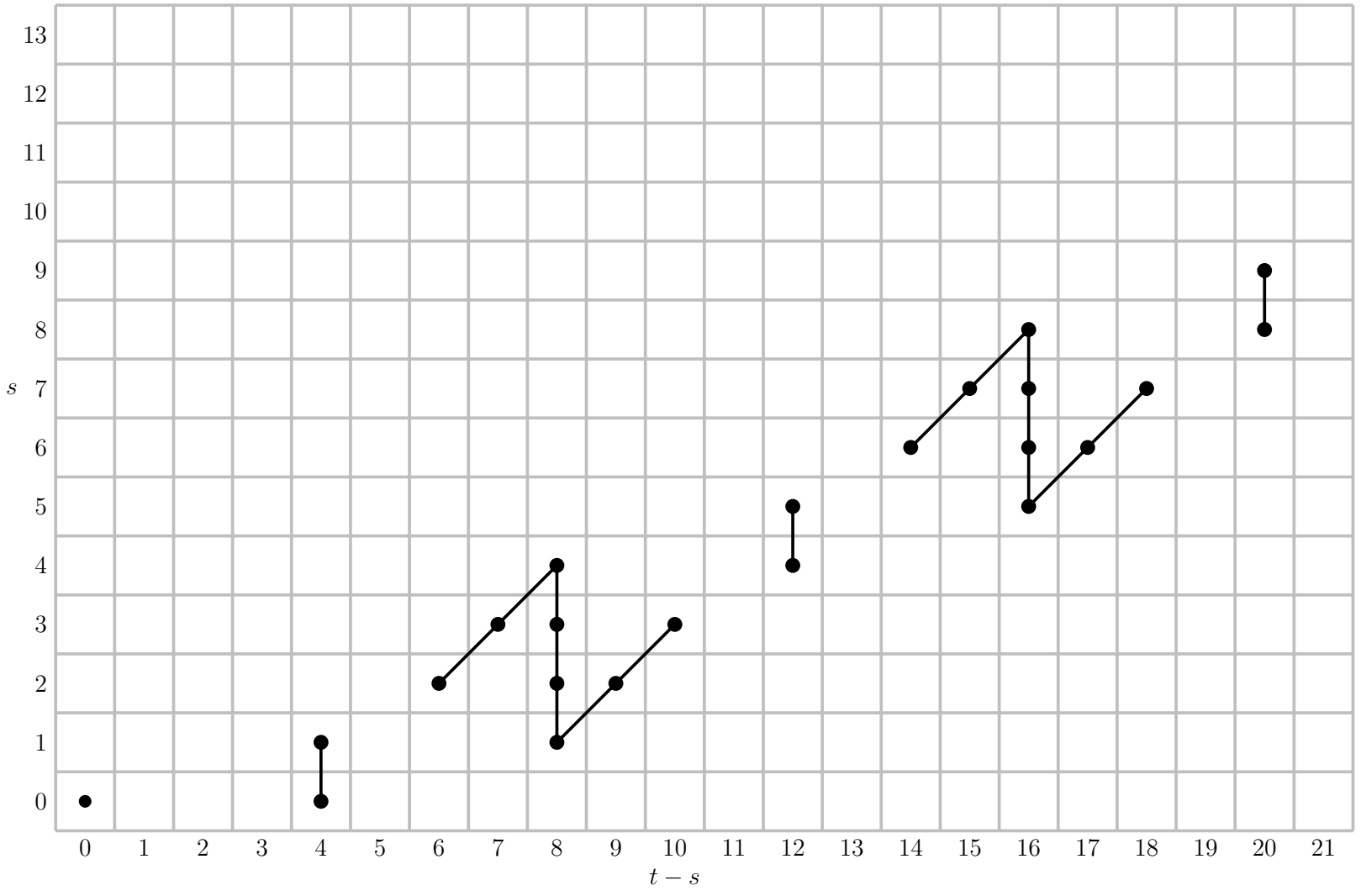
Long exact sequence in $Ext_{A(1)}$ from $B(6) \to B(8) \to \Sigma^4 B(4)$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

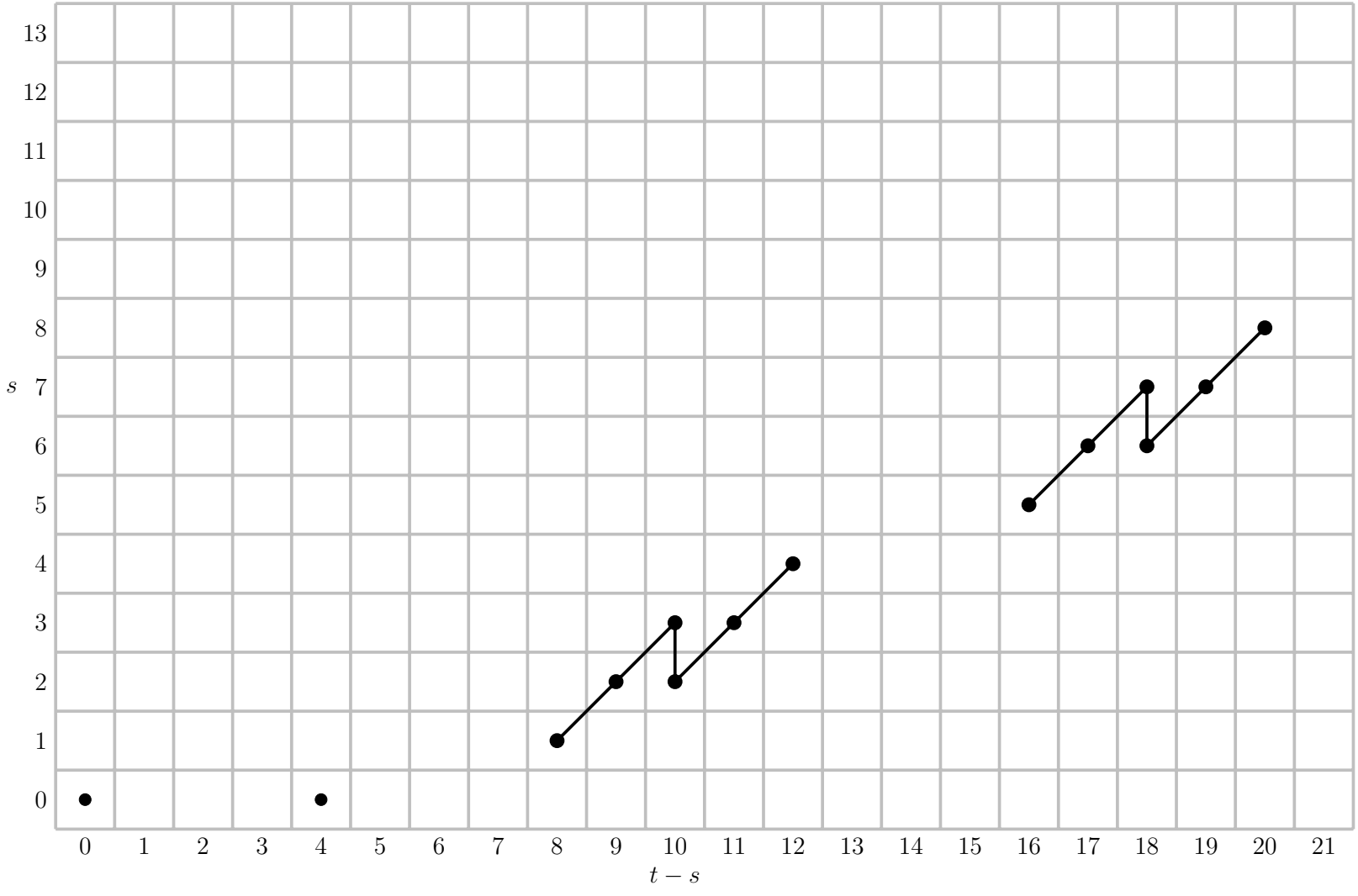
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$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(8))) \Rightarrow \mathrm{ko}_{t-s}(B(8))$$



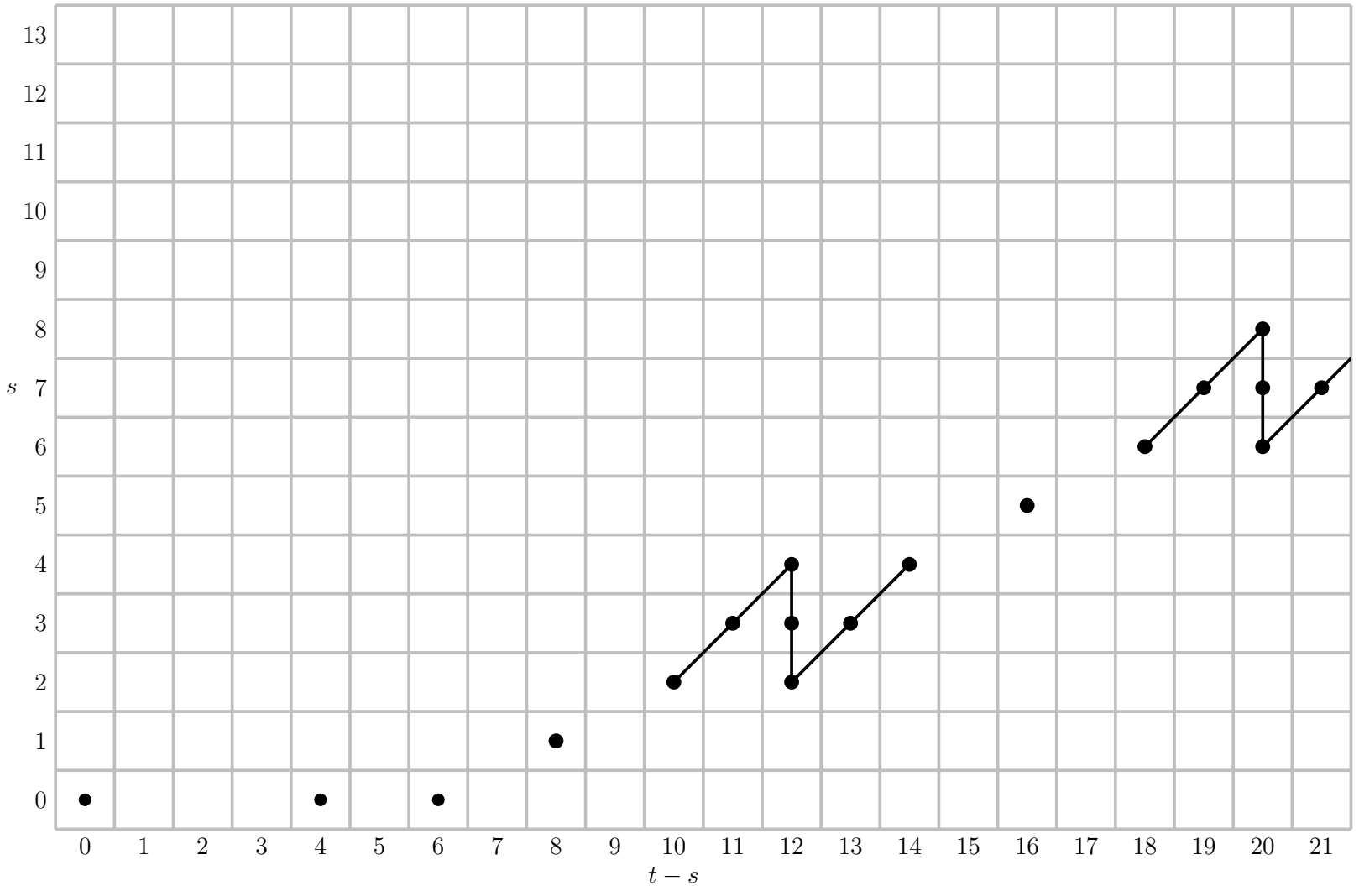
Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(10))) \Rightarrow \mathrm{ko}_{t-s}(B(10))$$



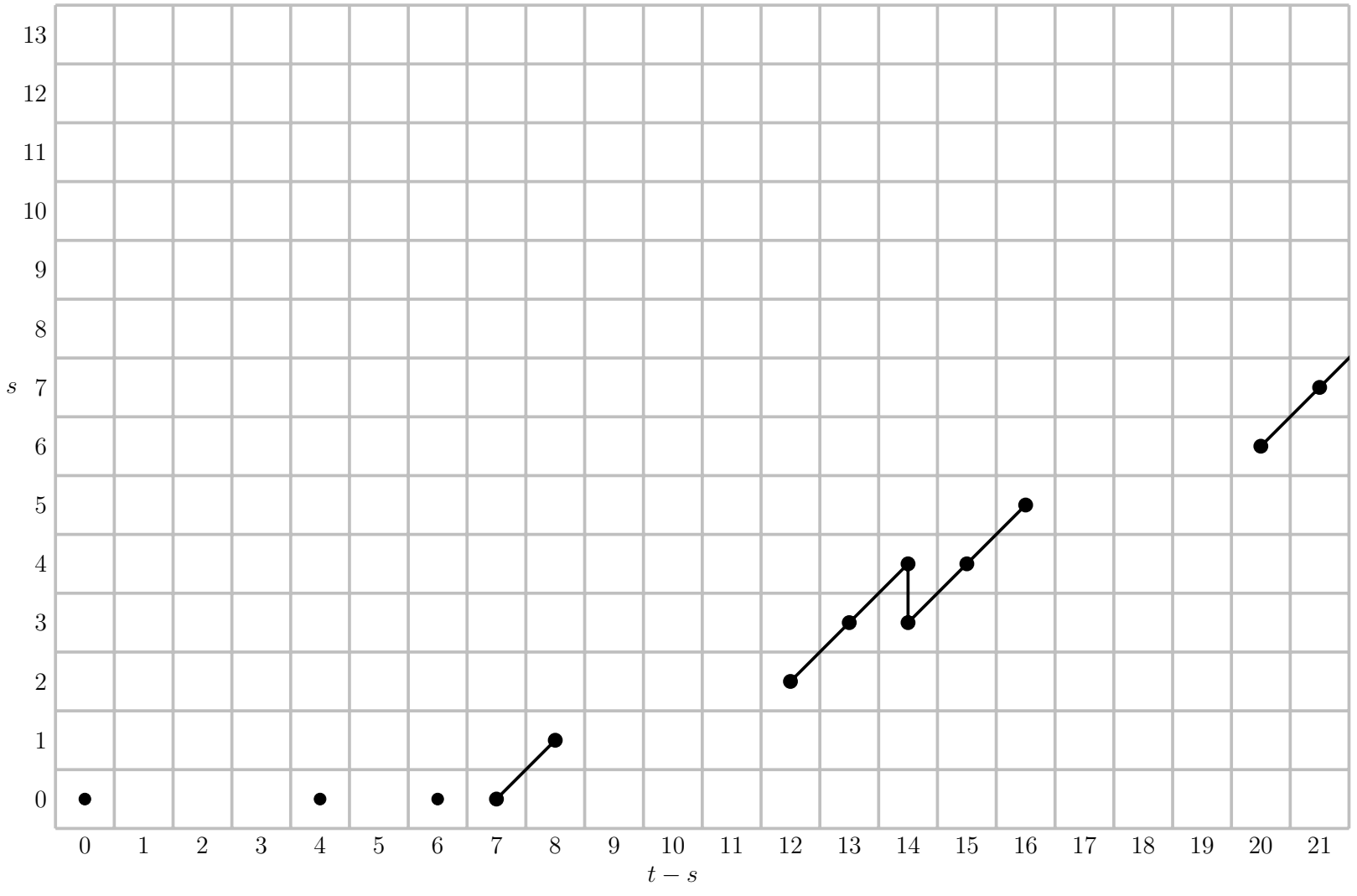
Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(12))) \Rightarrow \mathrm{ko}_{t-s}(B(12))$$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

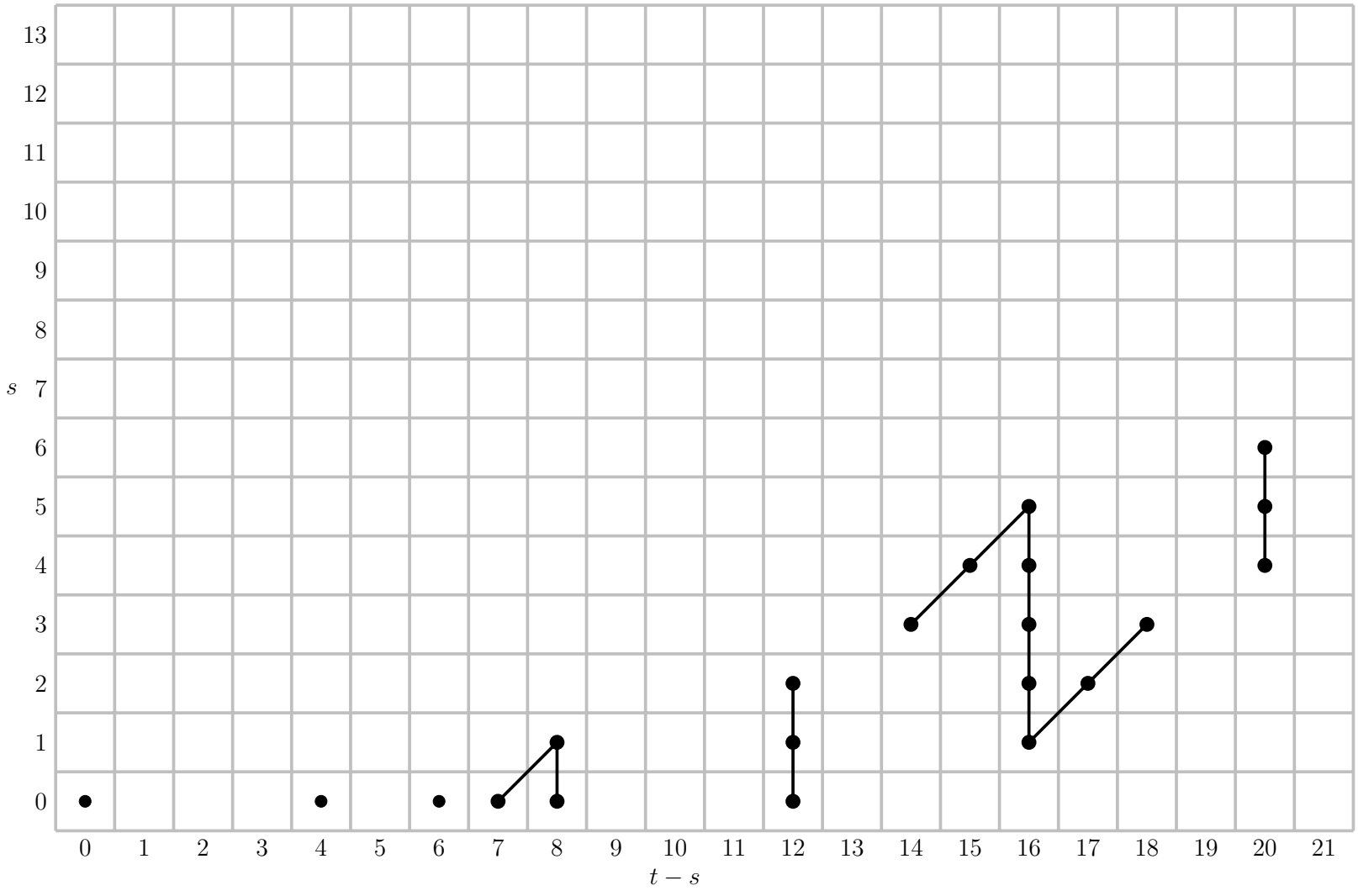
$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(14))) \Rightarrow \mathrm{ko}_{t-s}(B(14))$$



Adams S.S. for $\mathrm{ko}_{t-s}(B(2n))$

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$$\mathrm{Ext}^{s,t}_{A(1)}(F_2, H_*(B(16))) \Rightarrow \mathrm{ko}_{t-s}(B(16))$$



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Adams spectral sequence for ko

Dieudonné ring for ko

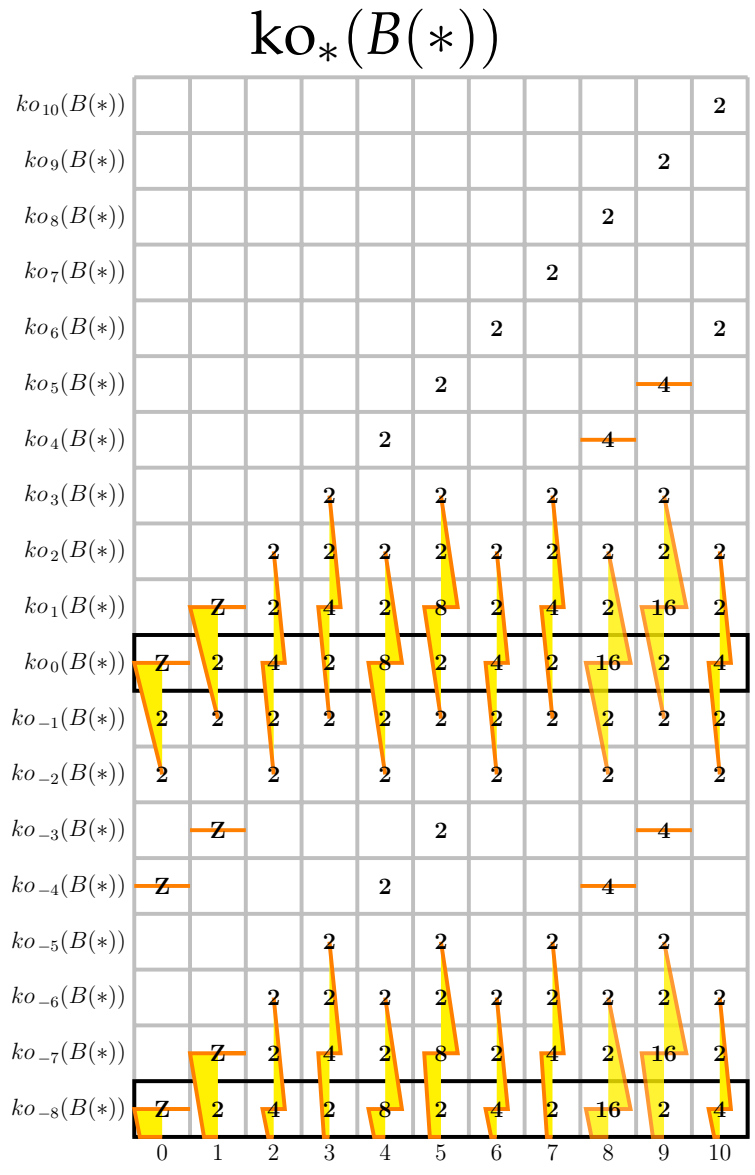
Margolis homology

Ext for $k(r)$

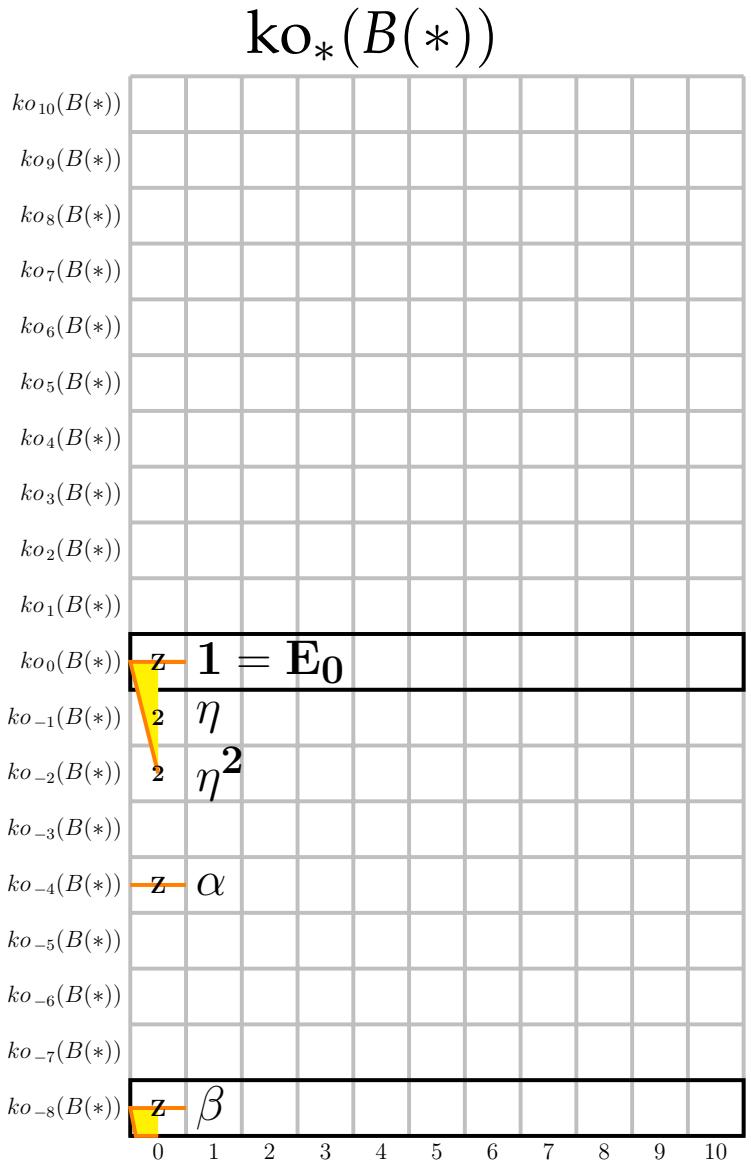
Dieudonné and Hopf ring results for $k(r)$

References

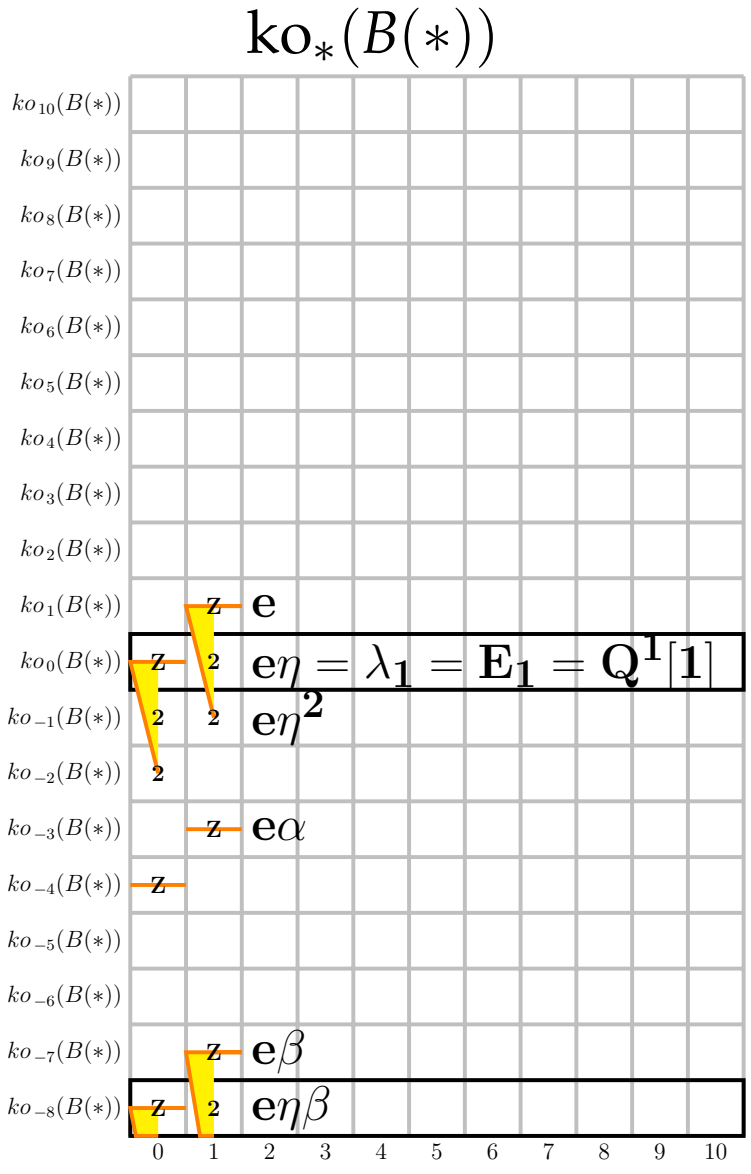
Mod 2 Dieudonné ring for ko



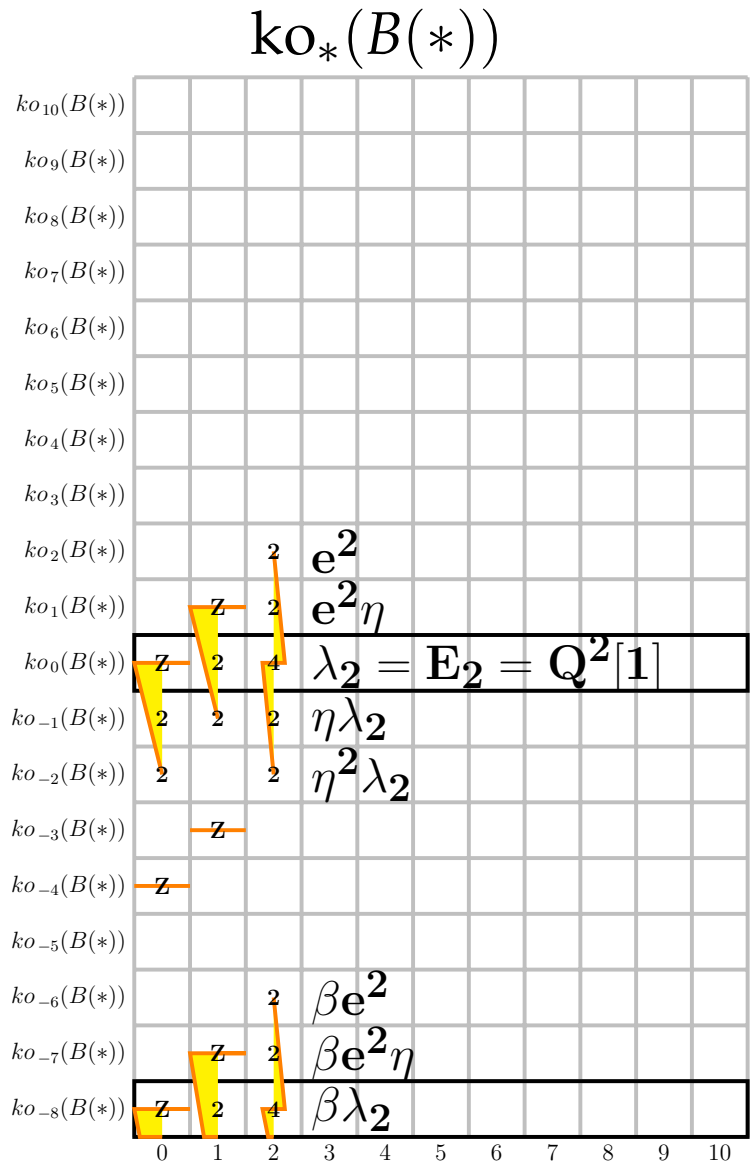
Mod 2 Dieudonné ring for *ko*



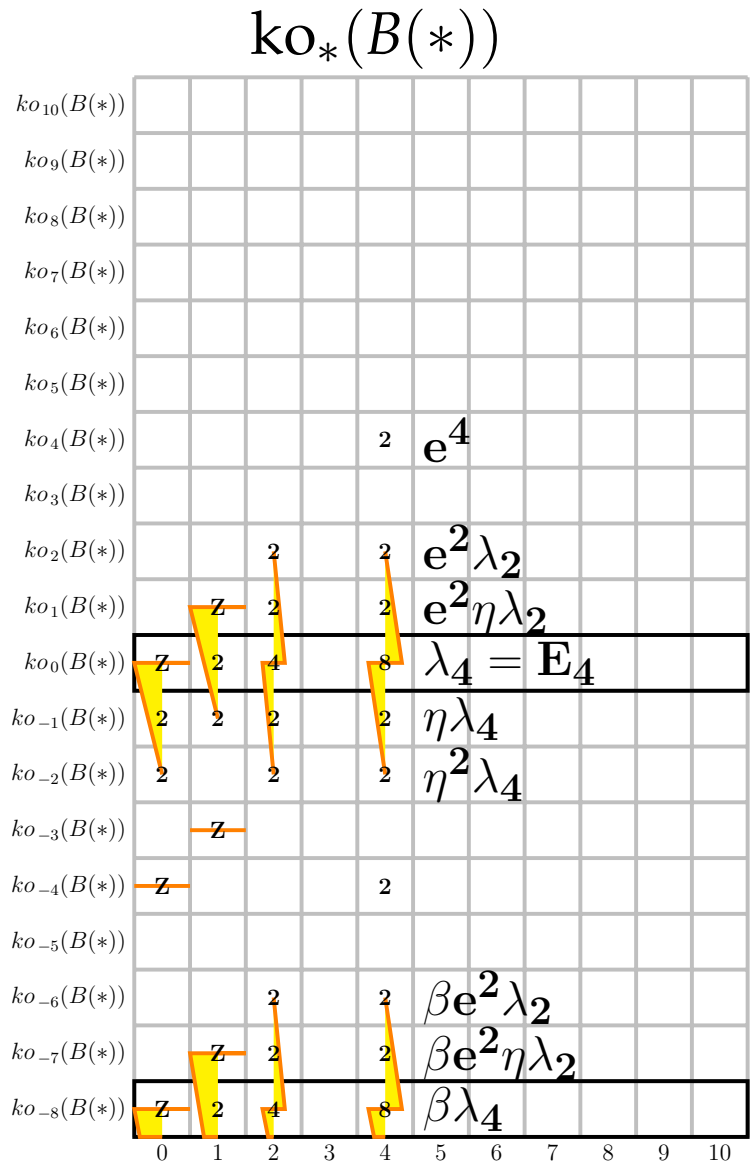
Mod 2 Dieudonné ring for *ko*



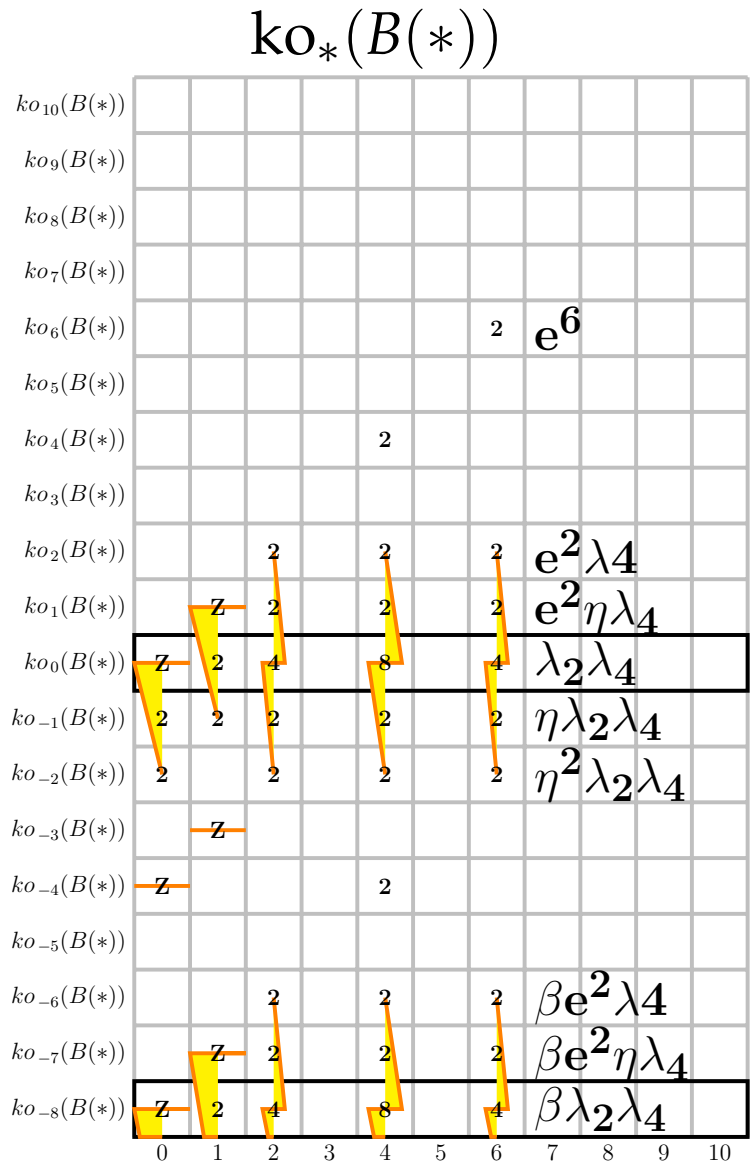
Mod 2 Dieudonné ring for *ko*



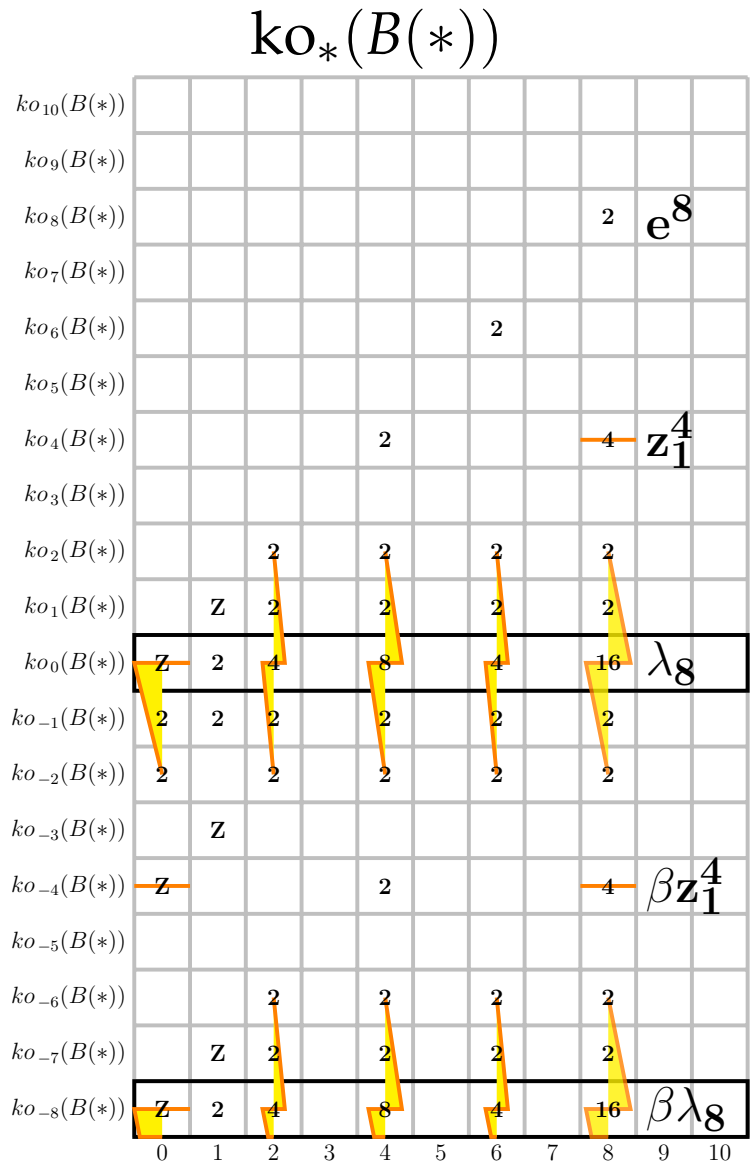
Mod 2 Dieudonné ring for *ko*



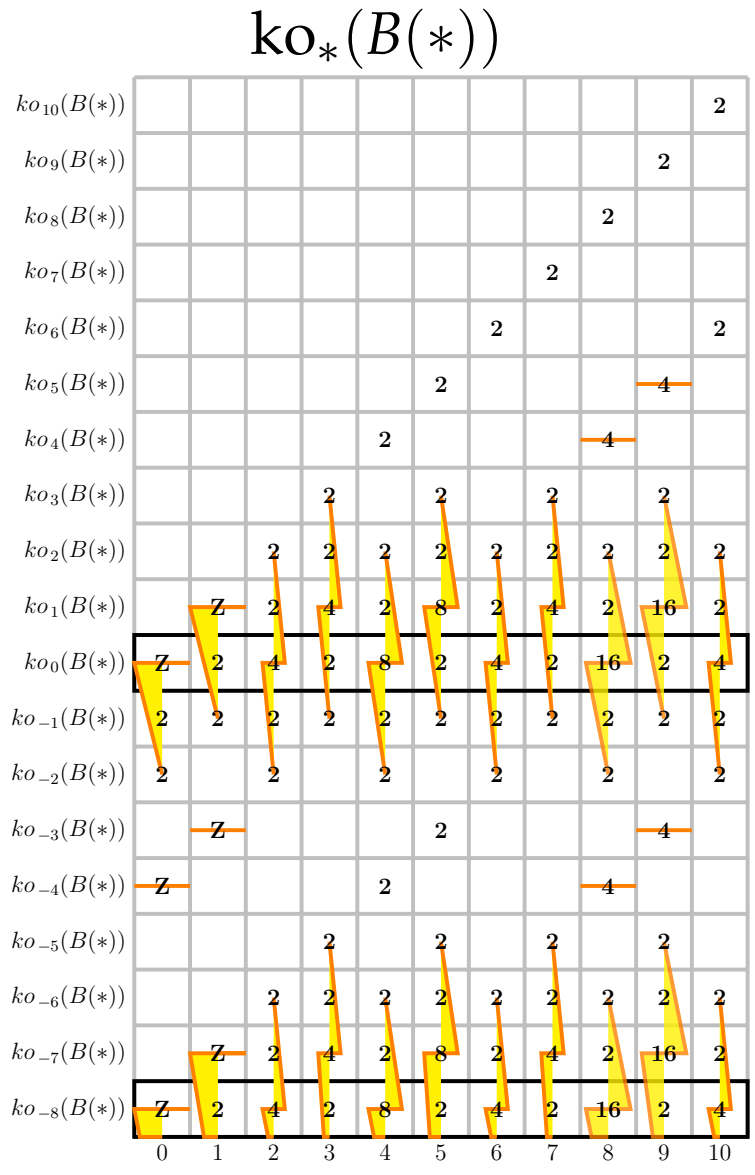
Mod 2 Dieudonné ring for *ko*



Mod 2 Dieudonné ring for *ko*



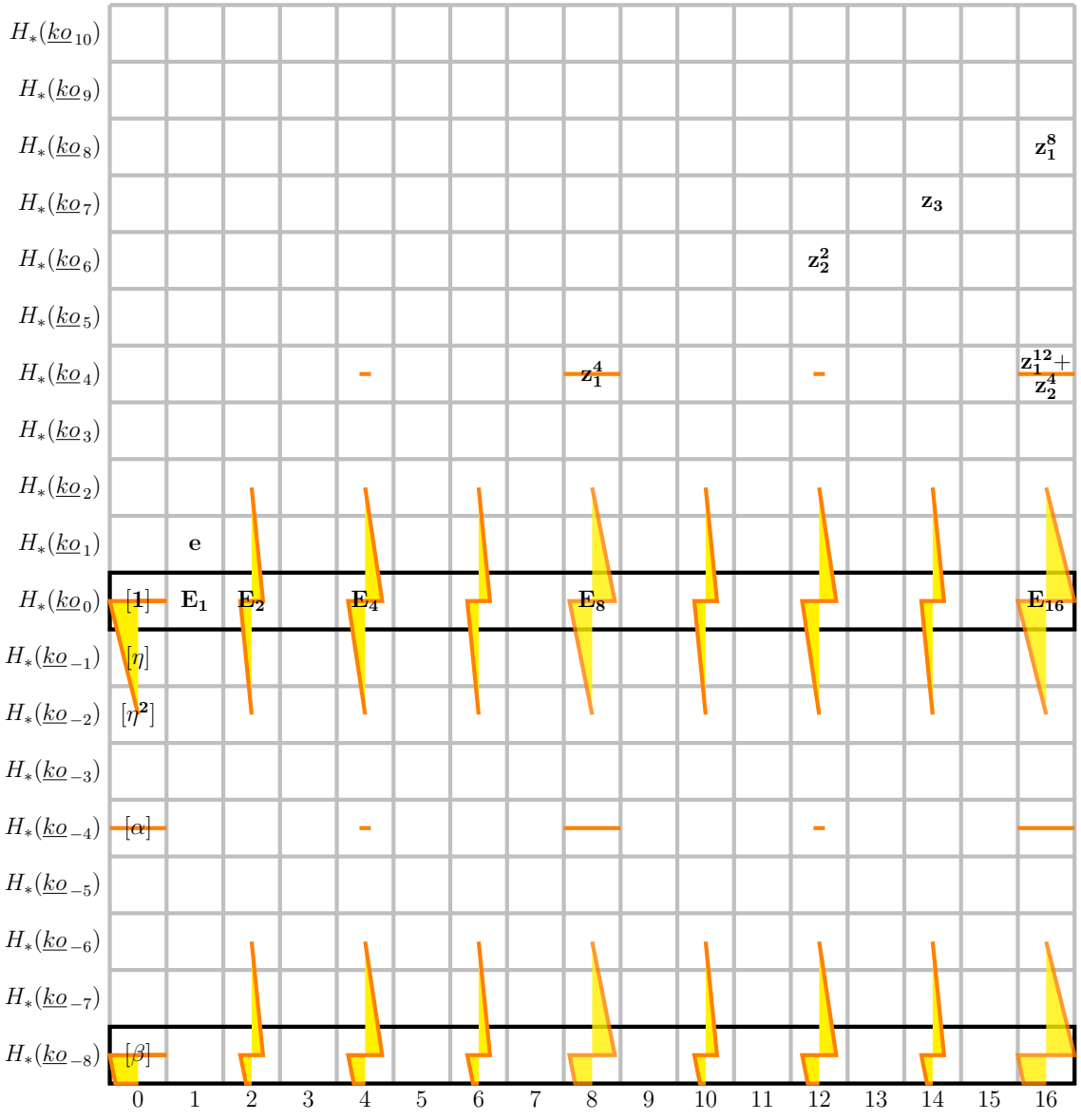
Mod 2 Dieudonné ring for ko



Mod 2 Hopf ring for *ko*

HOPE COLLEGE

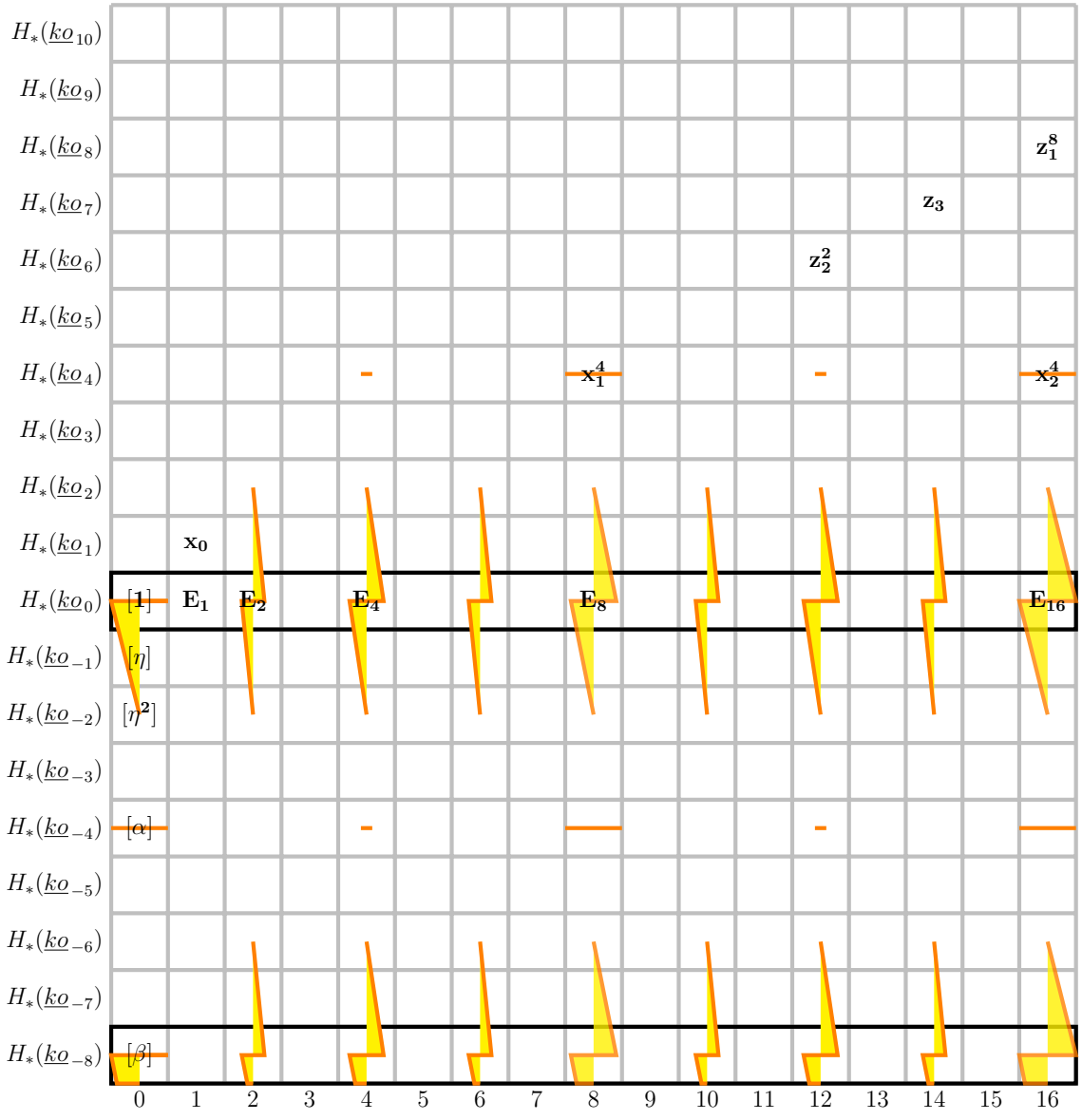
Conjectured generators for $H_*(\underline{ko}_*)$



Mod 2 Hopf ring for *ko*

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Conjectured generators for $H_*(\underline{ko}_*)$



Mod 2 Hopf ring for ko



Conjecture (Pearson)

The Hopf ring $H_*(\underline{\mathrm{ko}}_*)$ is generated by

- *stable generators $e^{-\infty}(H_*(\mathrm{ko}))$,*
- *zero space generators $E_{2^i} \in H_{2^i}(\underline{\mathrm{ko}}_0) = H_{2^i}(\mathbb{Z} \times BO)$, and*
- *homotopy generators $[\eta], [\alpha], [\beta]$ in $H_0(\underline{\mathrm{ko}}_k)$ for $k = -1, -4, -8$.*

Remarks:

- For the Dieudonné ring $D_*(H.(\underline{\mathrm{ko}}_*))$, I have essentially proved and published this result.
- Many of the relations in this Hopf ring can be read directly from the Ext charts $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, H_*(B(2n)))$
- Future work: it would be nice to have a presentation of this Hopf ring in terms of generators and relations (rework Cowen-Morton's and Strickland's results).

Mod 2 Hopf rings for ko and tmf

HOPE COLLEGE

Conjecture (Pearson)

The Hopf ring $H_*(\underline{\mathrm{tmf}}_*)$ *is generated by*

- *stable generators* $e^{-\infty}(H_*(\mathrm{tmf}))$,
- *zero space generators from* $H_*(\underline{\mathrm{tmf}}_0)$, *and*
- *homotopy generators from* $\pi_*^S(\mathrm{tmf})$.

Mod 2 Hopf rings for ko and tmf



Conjecture (Pearson)

The ring spectrum unit map $S \to \mathrm{tmf} \to \mathrm{ko}$ *induces Hopf algebra isomorphisms*

- $H_*(\underline{\mathrm{ko}}_0) = H_*(QS^0)/(N(E_i) \mid \alpha(i) > 1)$
- $H_*(\underline{\mathrm{tmf}}_0) = H_*(QS^0)/(N(E_i) \mid \alpha(i) > 2)$ *and possibly more relations*

where $N(E_i)$ *is a Newton polynomial in* $E_i = Q^i[1]$.

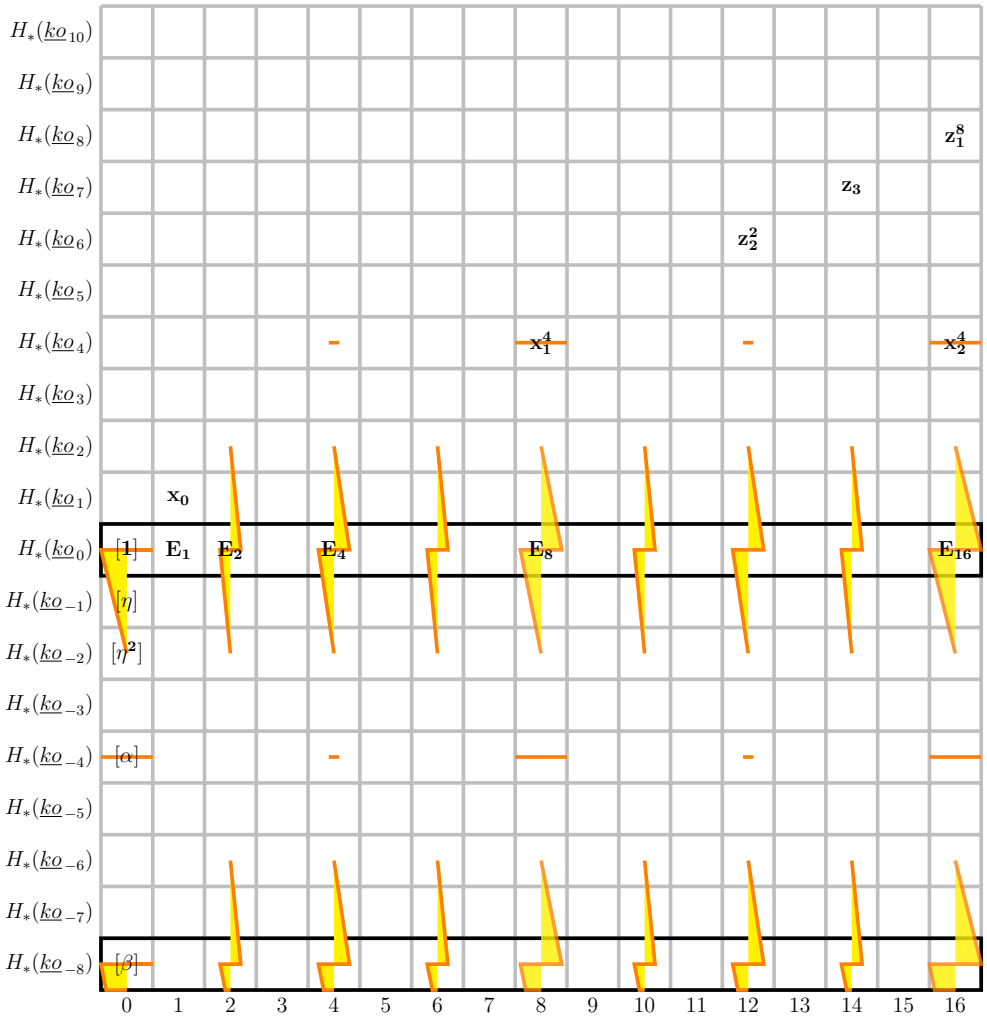
Highlights

- Dieudonné rings focus on $+$ and $\circ$, and use F and V to capture the essential $*$ product structure.
- We have identified the natural generators for $H_*(\underline{\mathrm{ko}}_*)$ coming from $\pi_*^S(\mathrm{ko})$, $H_*(\mathrm{ko})$, and $H_*(\underline{\mathrm{ko}}_0)$.
- We have defined the destabilization function and determined all the stable classes in $H_*(\underline{\mathrm{tmf}}_*)$ and $H_*(\underline{\mathrm{ko}}_*)$.
- For $k \geq 0$, we have observed that $H_*(\underline{\mathrm{tmf}}_k)$ and $H_*(\underline{\mathrm{ko}}_k)$ interpolate between $H_*(QS^k)$, for which the stabilization has a huge kernel, and $H_*(\underline{\mathrm{H}\mathbb{F}}_k)$, for which the stabilization has a tiny kernel.

Thank you!

HOPE COLLEGE

Any questions?



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Dieudonné rings
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Q_r Margolis homology



Theorem (Pearson)

The Q_0 Margolis homology of a Brown-Gitler module is

$$H(H_*(B(2n)), Q_0) = \begin{cases} \mathbb{F}_2\{1\} & \text{if } n = 0, \\ 0 & \text{if } n \geq 1. \end{cases}$$

For $r \geq 1$, the Q_r Margolis homology of a Brown-Gitler module is

$$H(H_*(B(2n)), Q_r)$$

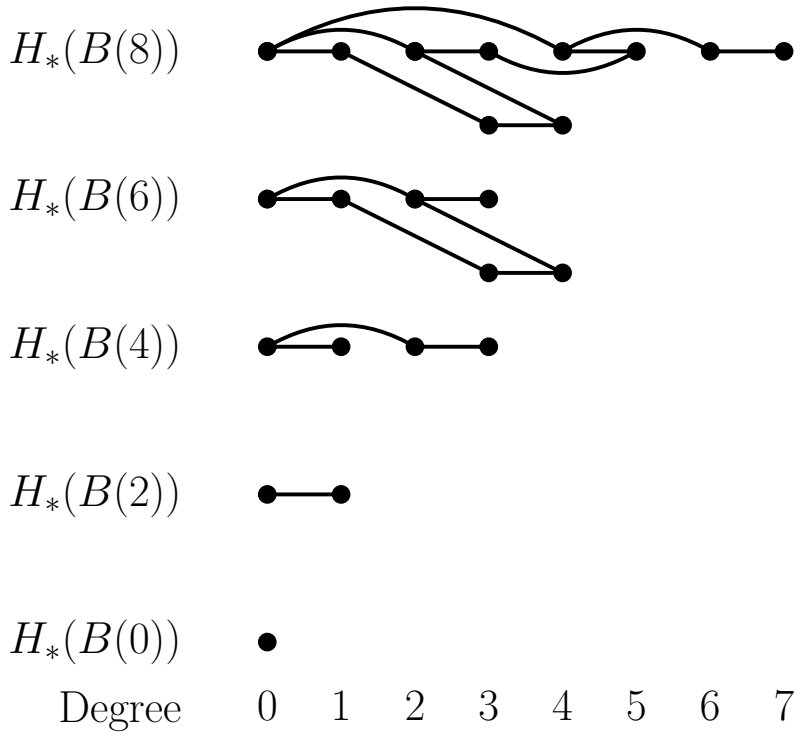
$$= \mathbb{F}_2 \left\{ \zeta^I \in T_2(\zeta_1, \ldots, \zeta_r) \otimes T_1(\zeta_{r+1}^2, \zeta_{r+2}^2, \ldots) \mid n - 2^r \leq wt_\zeta(\zeta^I) \leq n \right\}$$

Remark: $T_m(x_1, x_2, \ldots) = P(x_1, x_2, \ldots)/(x_1^{2^m}, x_2^{2^m}, \ldots)$ denotes a truncated polynomial algebra.

Q_0 Margolis homology

Example

$H(H_*(B(0)), Q_0) = \mathbb{F}_2\{1\}$ and $H(H_*(B(2n)), Q_0) = 0$ for $n \geq 1$.

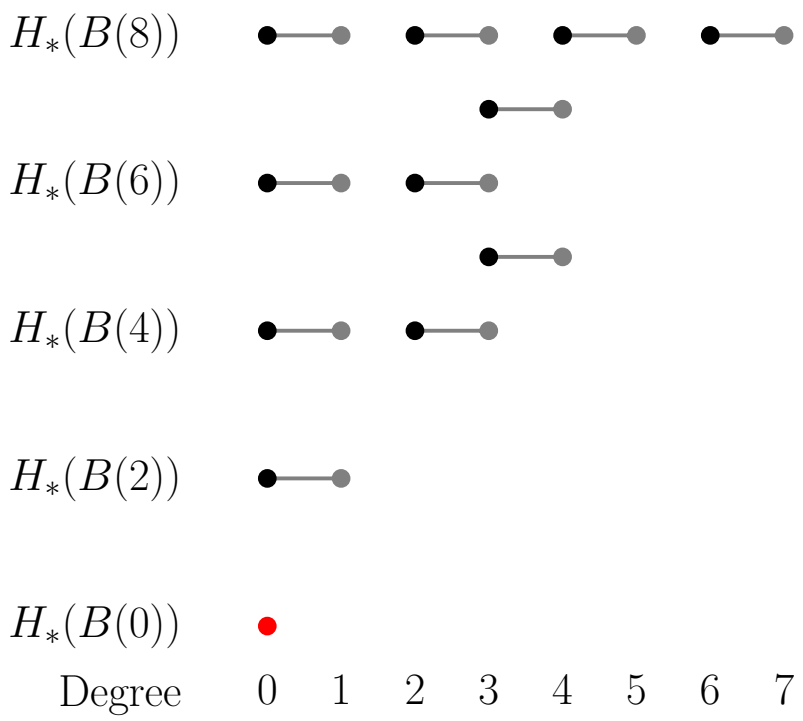


Brown-Gitler Modules ($\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4$ action shown)

Q_0 Margolis homology

Example

$H(H_*(B(0)), Q_0) = \mathbb{F}_2\{1\}$ and $H(H_*(B(2n)), Q_0) = 0$ for $n \geq 1$.

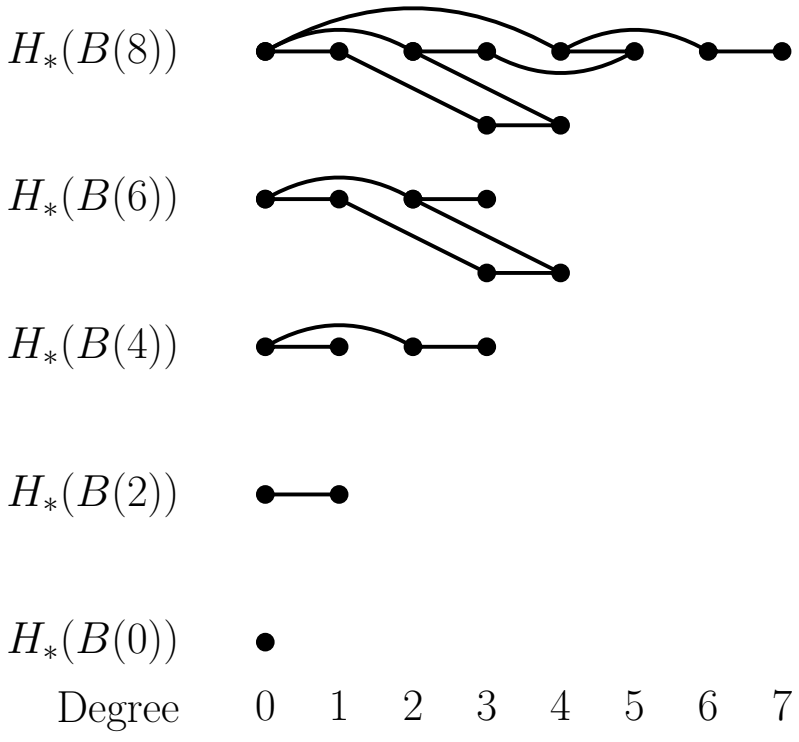


Non-zero Q_0 Margolis Homology (red)

Q_1 Margolis homology

Example

$H(H_*(B(n)), Q_1) = \mathbb{F}_2\{\zeta^I \in T_2(\zeta_1) \otimes T_1(\zeta_2^2, \zeta_3^2, \ldots) \mid n - 2 < wt_\zeta(\zeta^I) \leq n\}$.

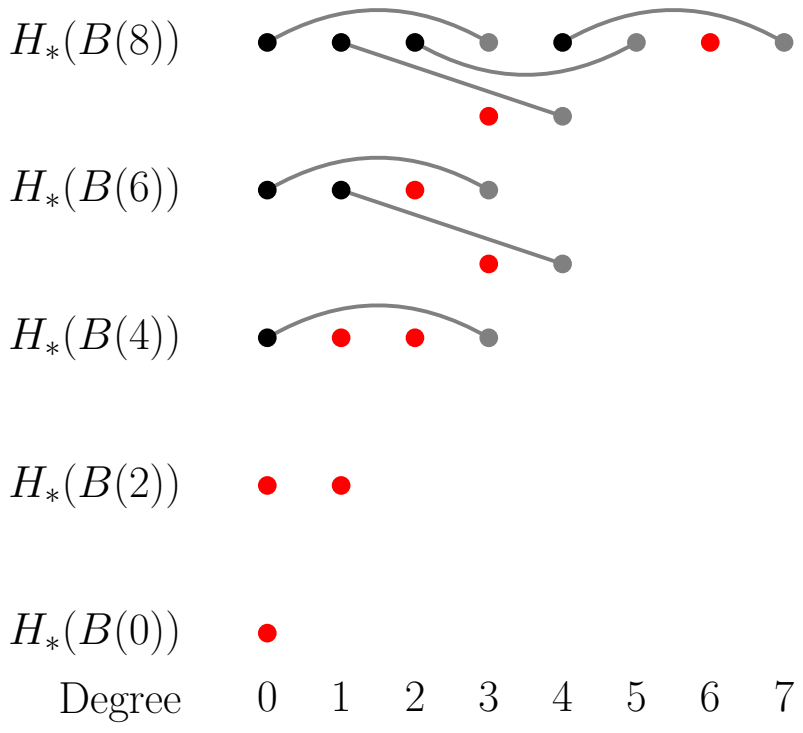


Brown-Gitler Modules (Sq^1, Sq^2, Sq^4 action shown)

Q_1 Margolis homology

Example

$H(H_*(B(n)), Q_1) = \mathbb{F}_2\{\zeta^I \in T_2(\zeta_1) \otimes T_1(\zeta_2^2, \zeta_3^2, \ldots) \mid n - 2 < wt_\zeta(\zeta^I) \leq n\}$.



Non-zero Q_1 Margolis Homology (red)

Problem

Calculate the Q_r Margolis homology of $H_*(B(n))$ for all primes $p \geq 3$.

I think I've done this calculation already, but I'm quite certain nobody has published it and that it would be low-hanging fruit.

Outline

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Hopf rings

Dieudonné rings

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Adams spectral sequence for ko

Dieudonné ring for ko

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Ext for $k(r)$

Dieudonné and Hopf ring results for $k(r)$

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Adams S.S. for $k(1)_*B(*)$

HOPE COLLEGE

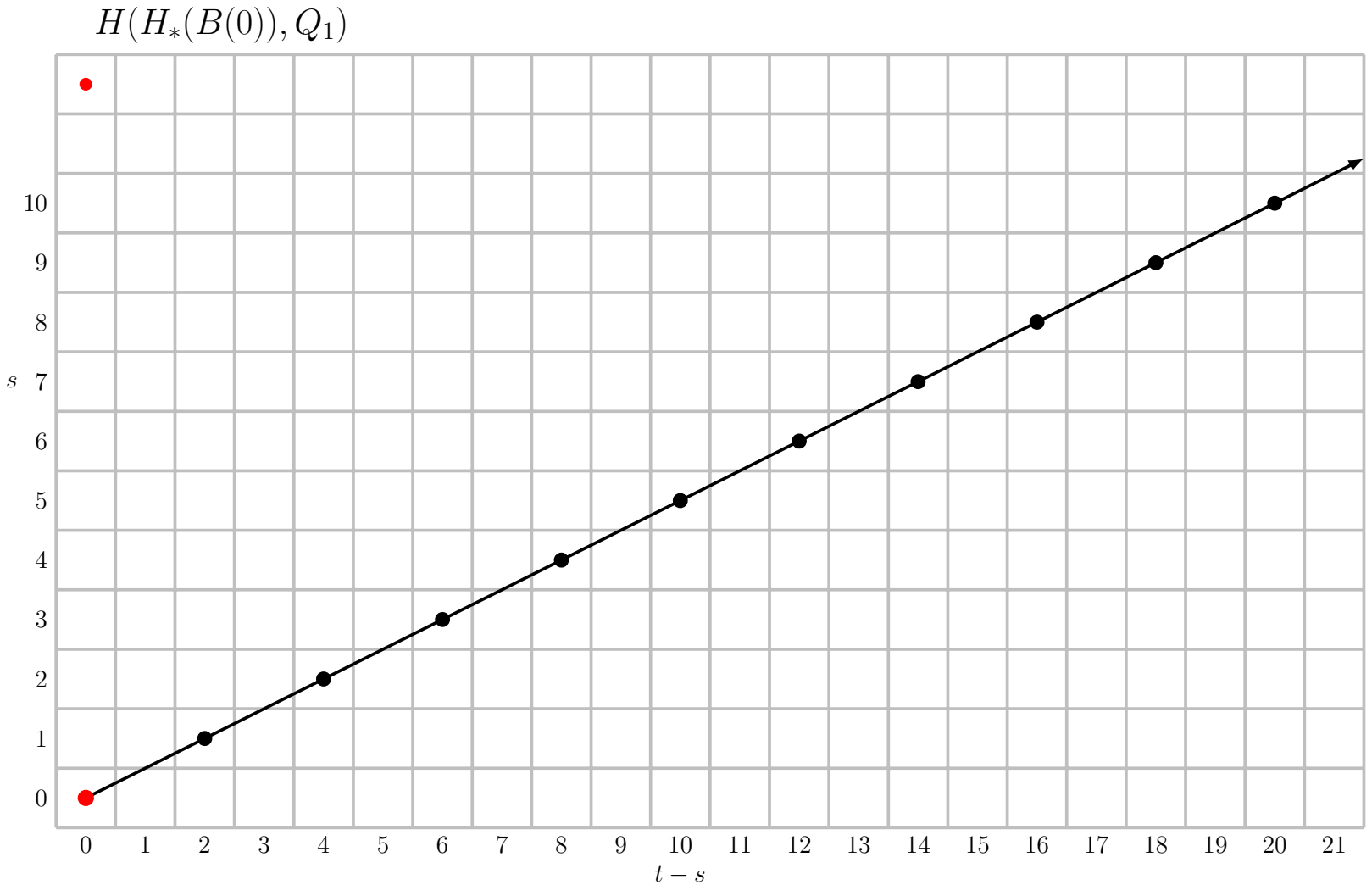


Theorem (Pearson)

Let $\mathrm{Ext}^{*,*}_{E(Q_r)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[a_r]$, *where* a_r *detects* $v_r \in k(r)_* = \mathbb{F}_2[v_r]$. *There is a one-to-one correspondence of* $\mathbb{F}_2$*-modules between classes in* $\mathrm{Ext}^{*,*}_{E(Q_r)}(\mathbb{F}_2, H_*(B(2n)))$ *that support nonzero multiplication by* $\mathbb{F}[a_r]$ *and nonzero classes in the Margolis homology* $H(H_*(B(2n)), Q_r)$.

Adams S.S. for $k(1)_*B(*)$

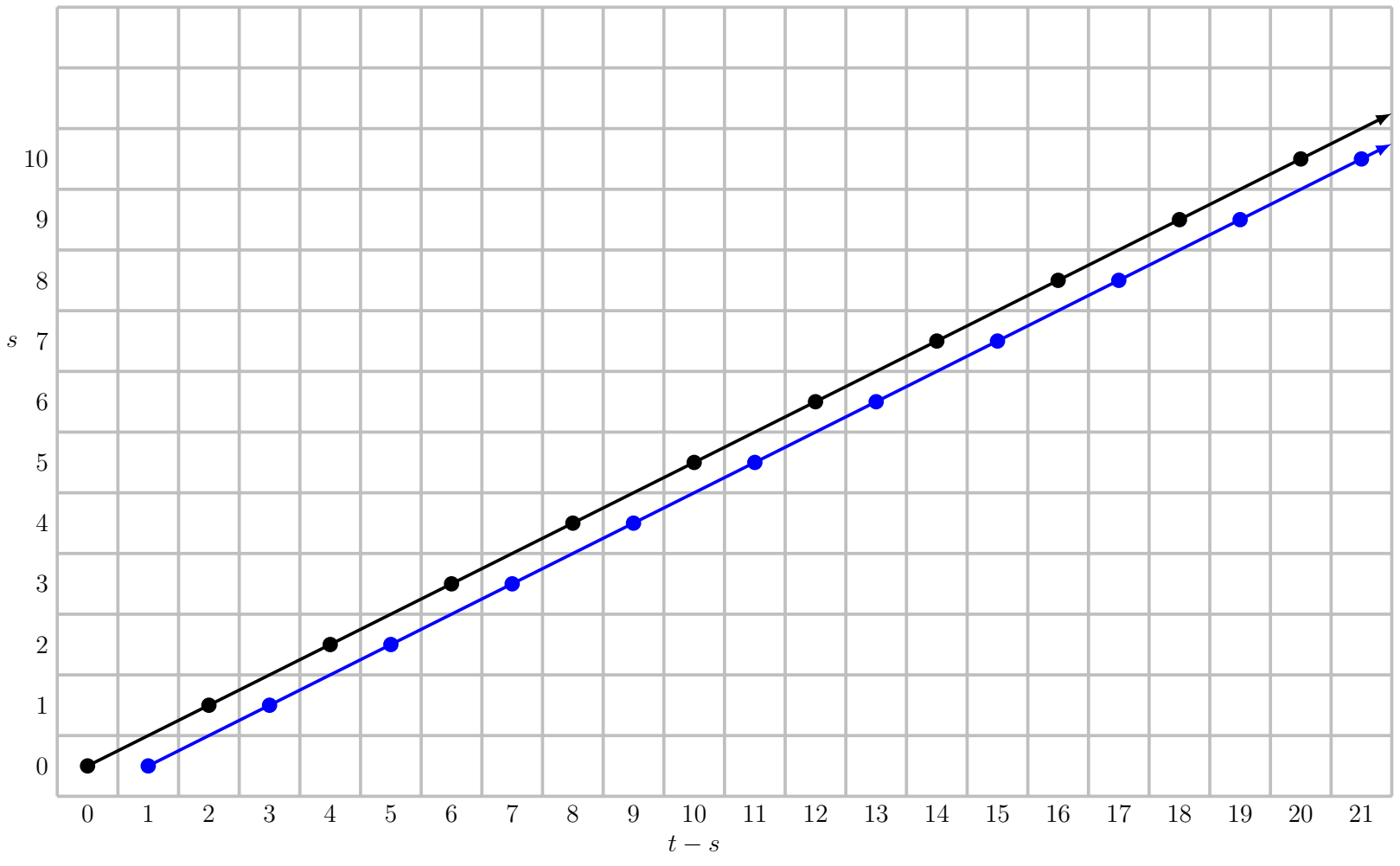
$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(0))) \Rightarrow \mathrm{k}(1)_{t-s}(B(0))$$



Adams S.S. for $k(1)_*B(*)$

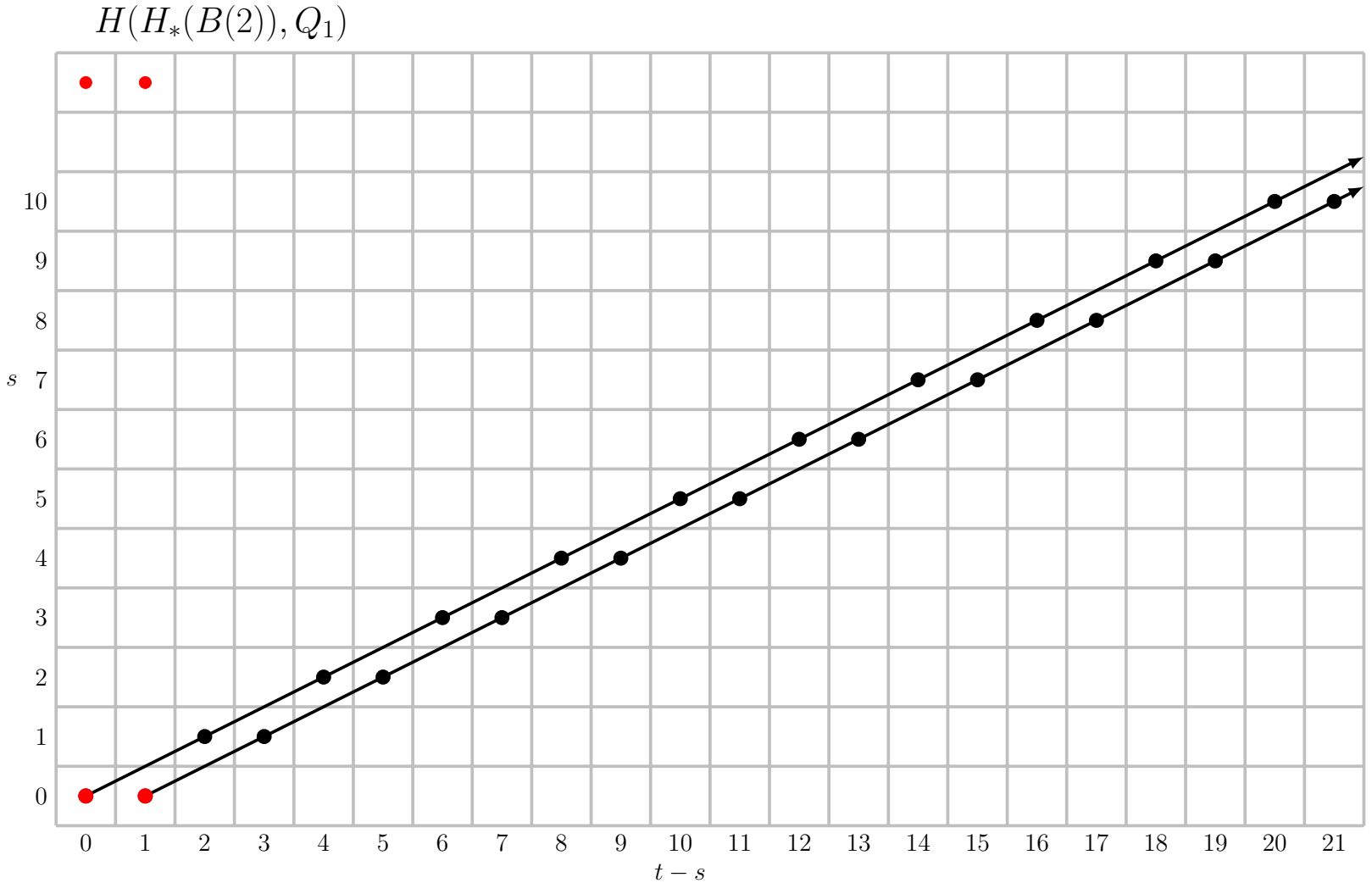
$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(2))) \Rightarrow \mathrm{k}(1)_{t-s}(B(2))$$

$B(0) \to B(2) \to \Sigma^1 B(1)$



Adams S.S. for $k(1)_*B(*)$

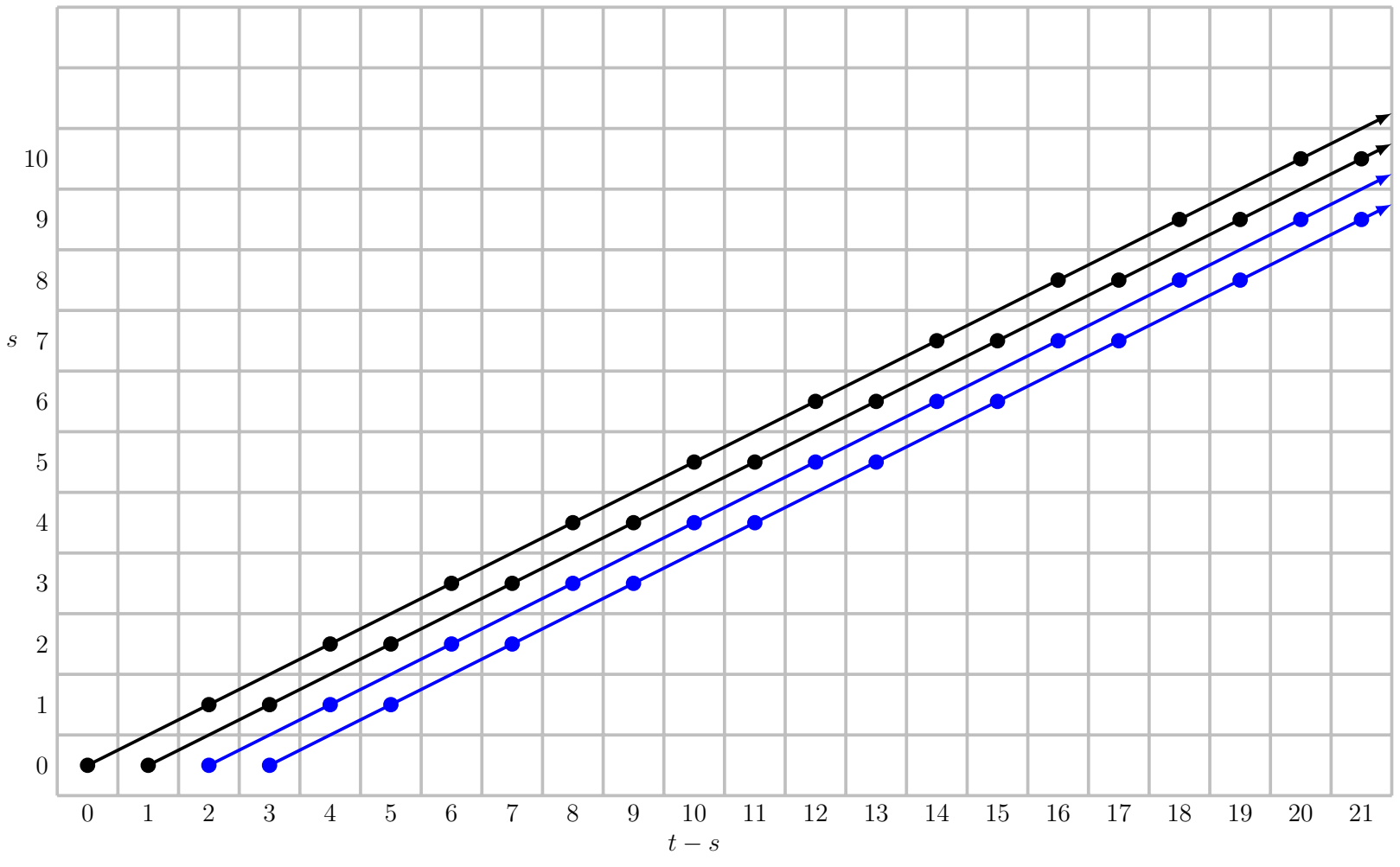
$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(2))) \Rightarrow \mathrm{k}(1)_{t-s}(B(2))$$



Adams S.S. for $k(1)_*B(*)$

$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(4))) \Rightarrow \mathrm{k}(1)_{t-s}(B(4))$$

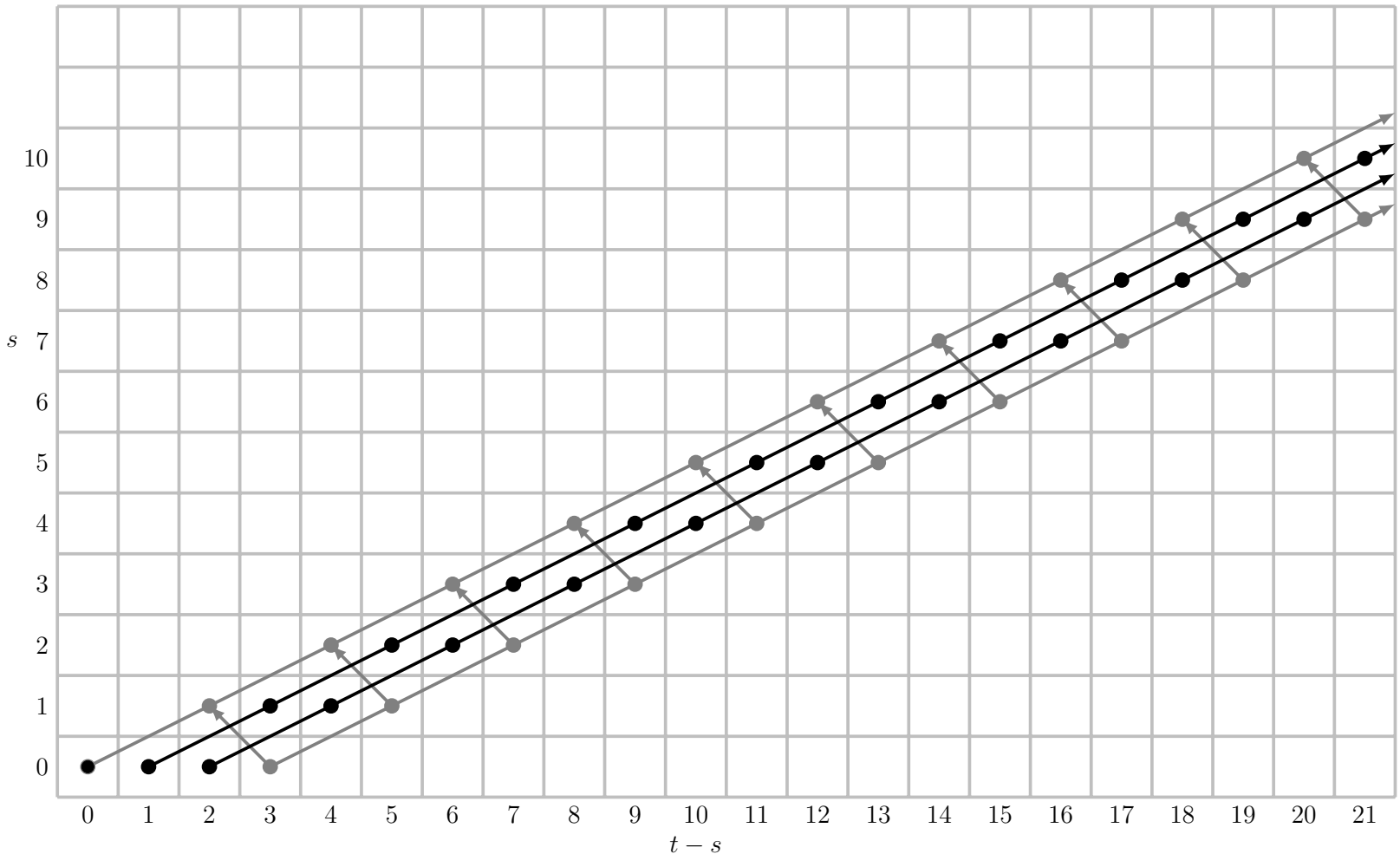
$B(2) \to B(4) \to \Sigma^2 B(2)$



Adams S.S. for $k(1)_*B(*)$

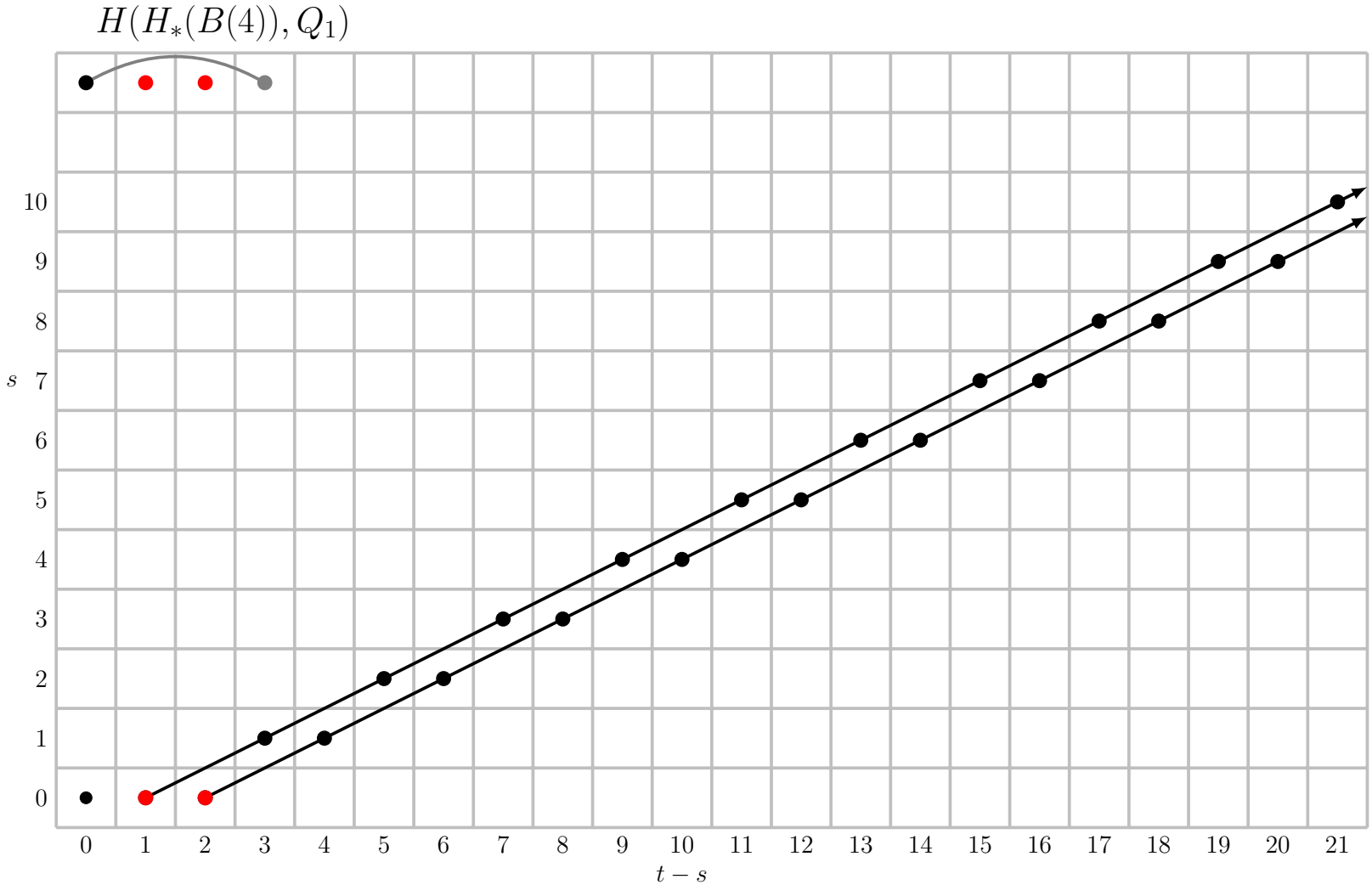
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$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(4))) \Rightarrow \mathrm{k}(1)_{t-s}(B(4))$$



Adams S.S. for $k(1)_*B(*)$

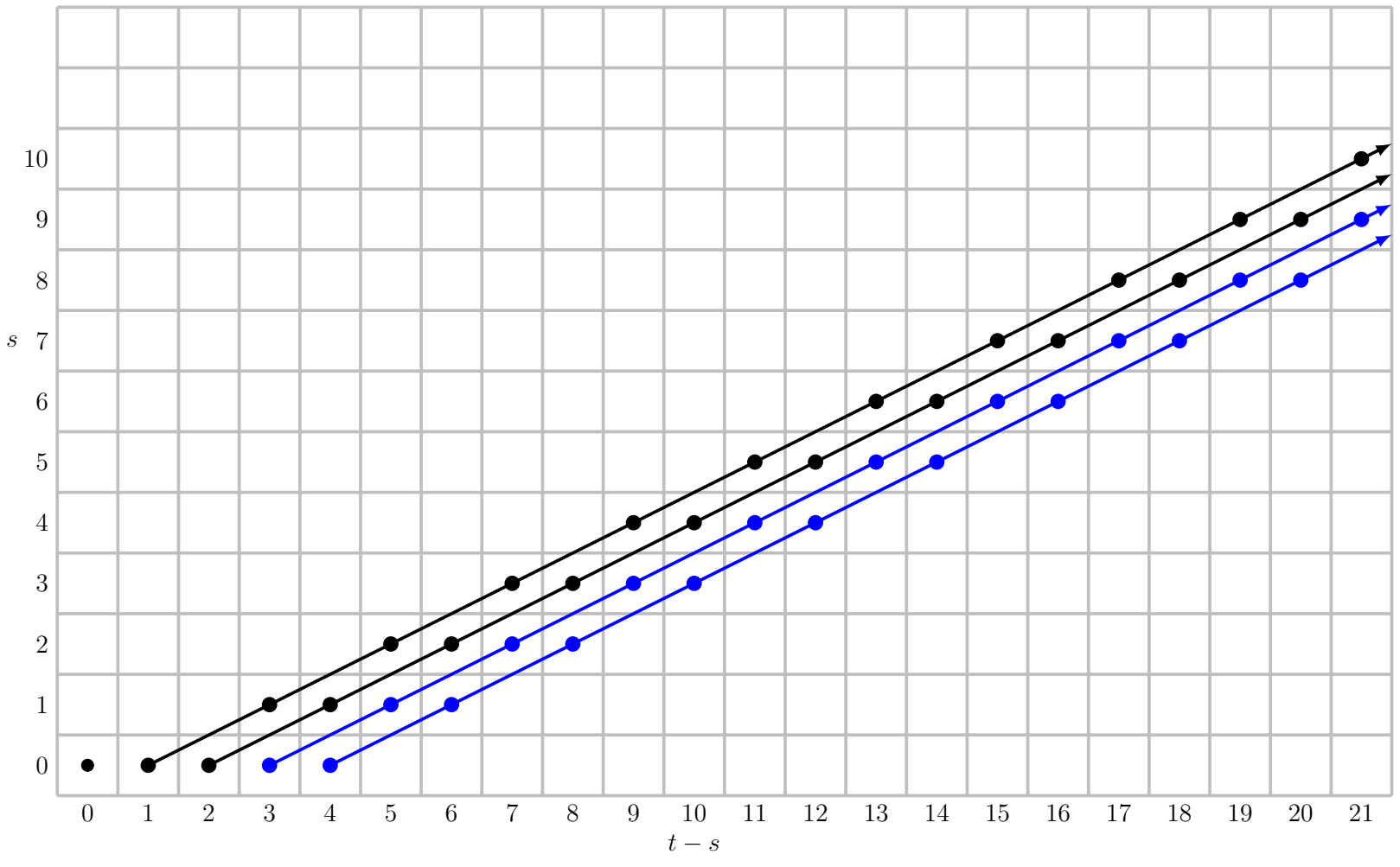
$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(4))) \Rightarrow \mathrm{k}(1)_{t-s}(B(4))$$



Adams S.S. for $k(1)_*B(*)$

$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(6))) \Rightarrow \mathrm{k}(1)_{t-s}(B(6))$$

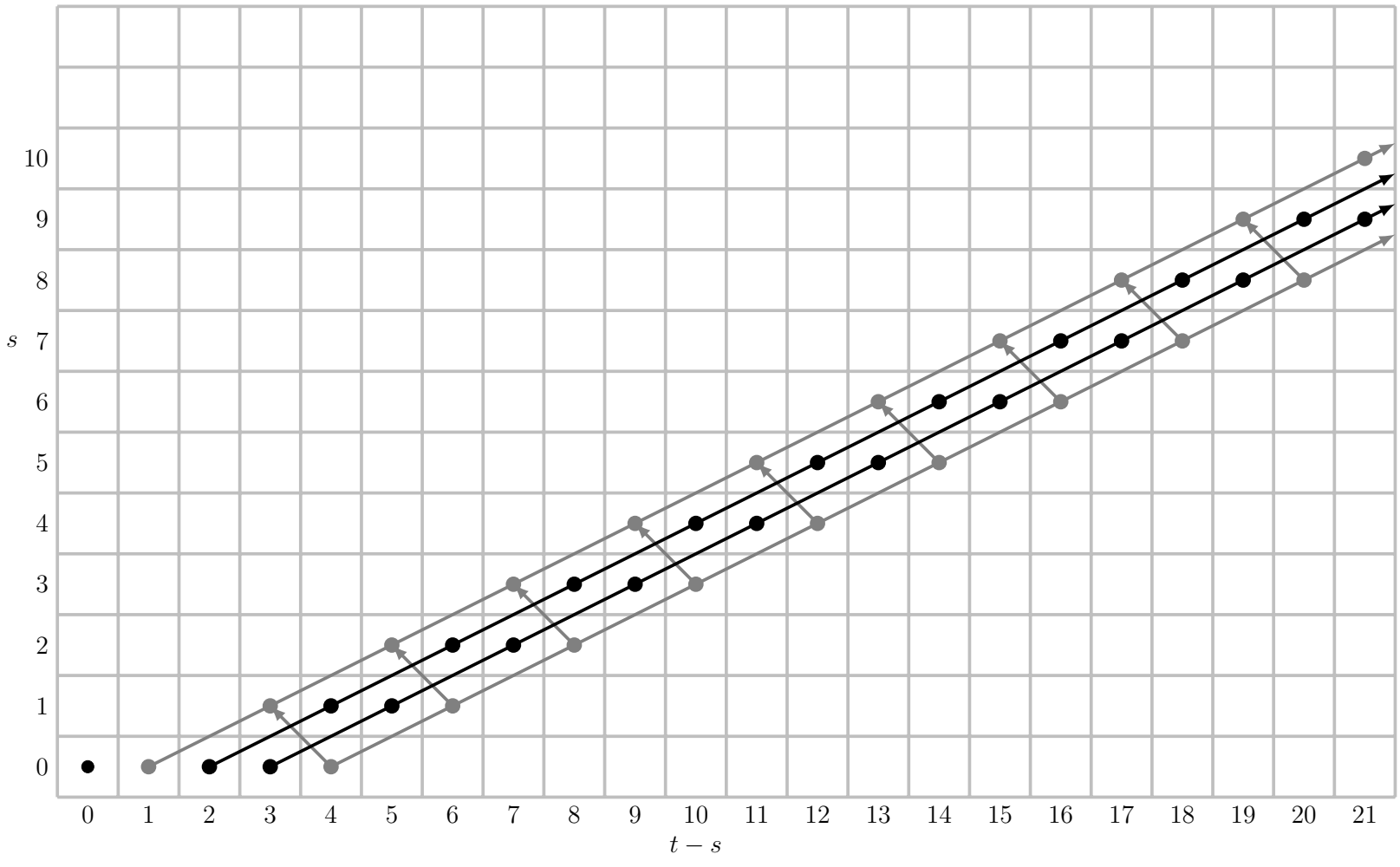
$B(4) \to B(6) \to \Sigma^3 B(3)$



Adams S.S. for $k(1)_*B(*)$

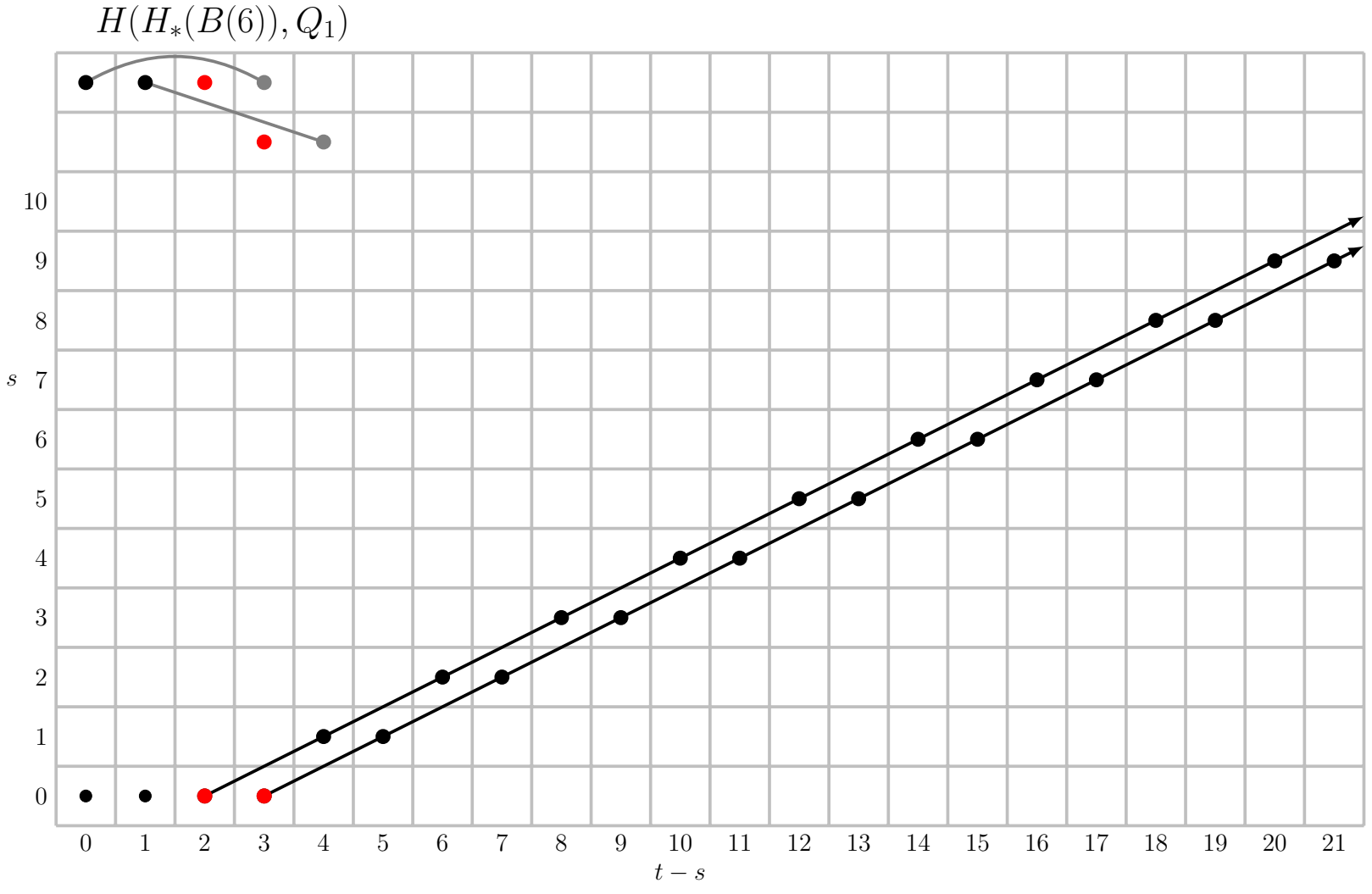
HOPE COLLEGE

$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(6))) \Rightarrow \mathrm{k}(1)_{t-s}(B(6))$$



Adams S.S. for $k(1)_*B(*)$

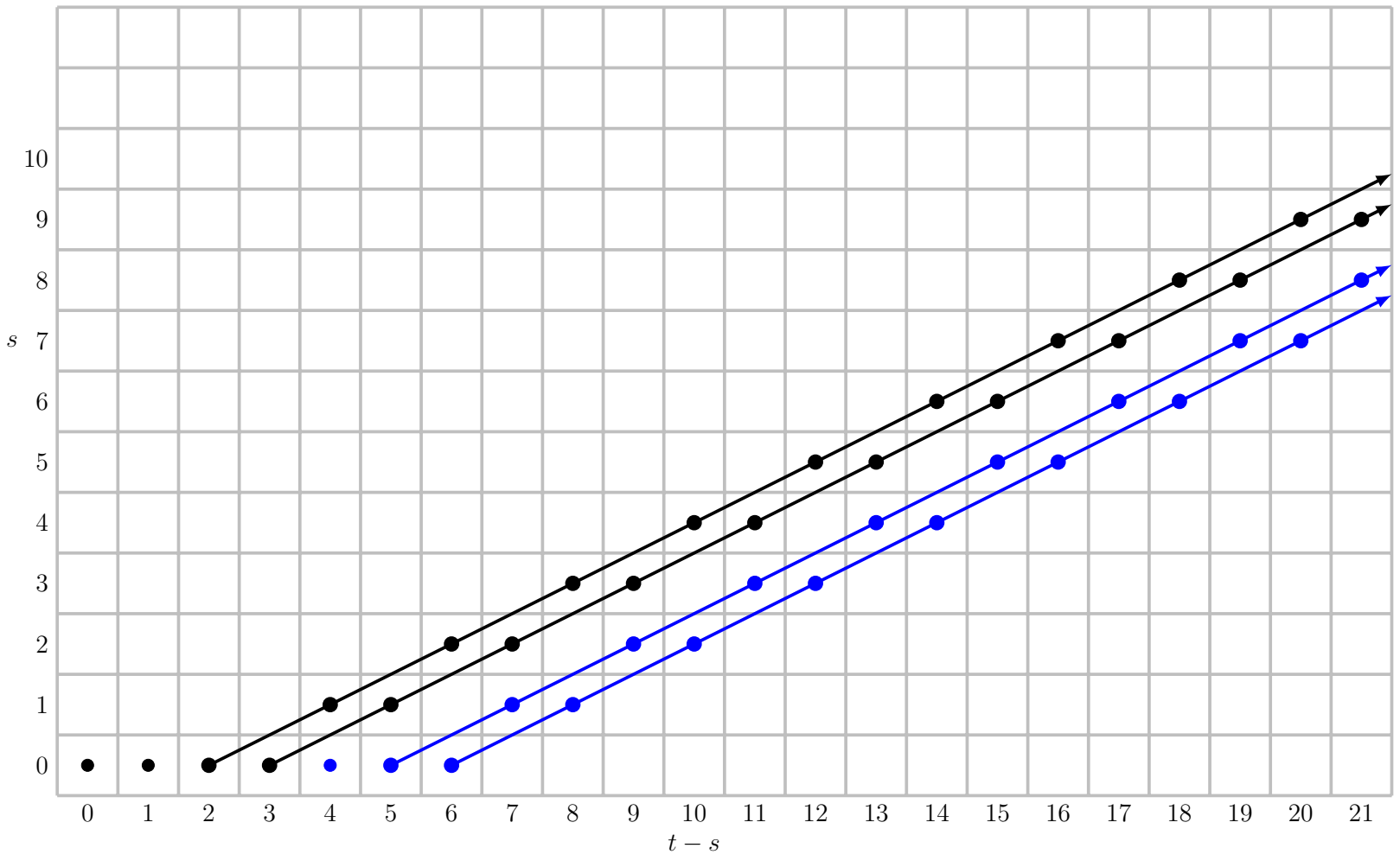
$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(6))) \Rightarrow \mathrm{k}(1)_{t-s}(B(6))$$



Adams S.S. for $k(1)_*B(*)$

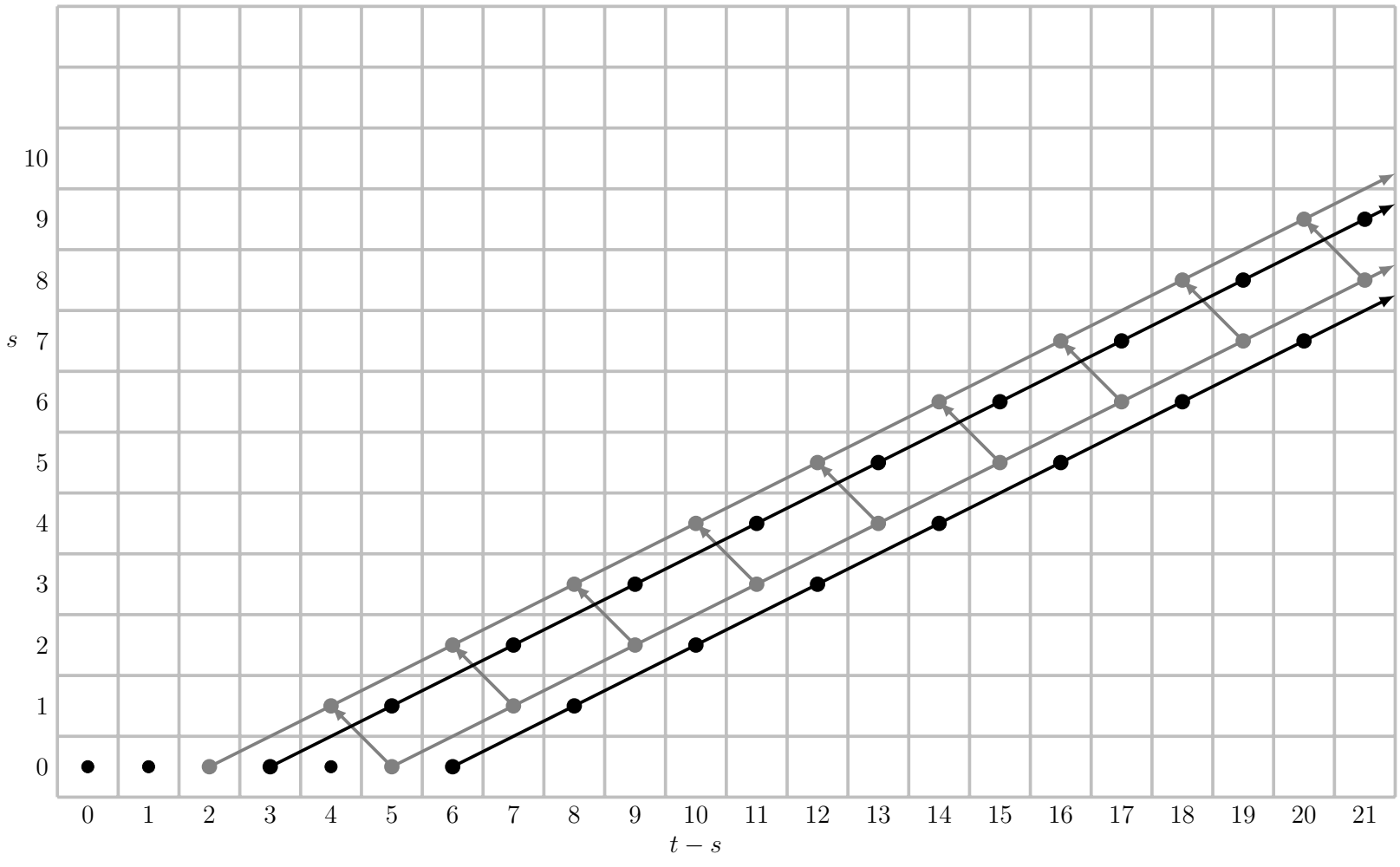
$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(8))) \Rightarrow \mathrm{k}(1)_{t-s}(B(8))$$

$B(6) \to B(8) \to \Sigma^4 B(4)$



Adams S.S. for $k(1)_*B(*)$

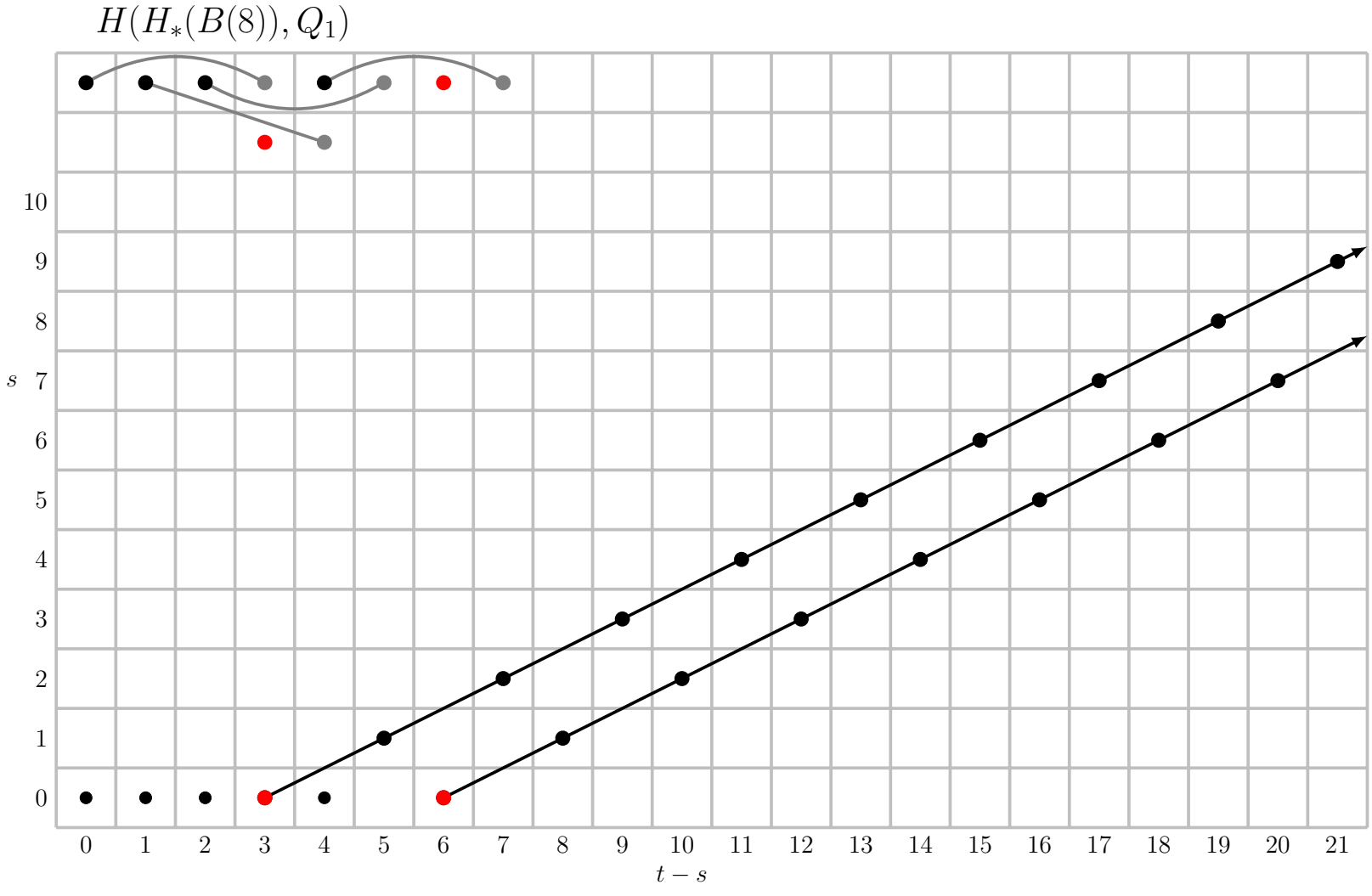
$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(8))) \Rightarrow \mathrm{k}(1)_{t-s}(B(8))$$



Adams S.S. for $k(1)_*B(*)$

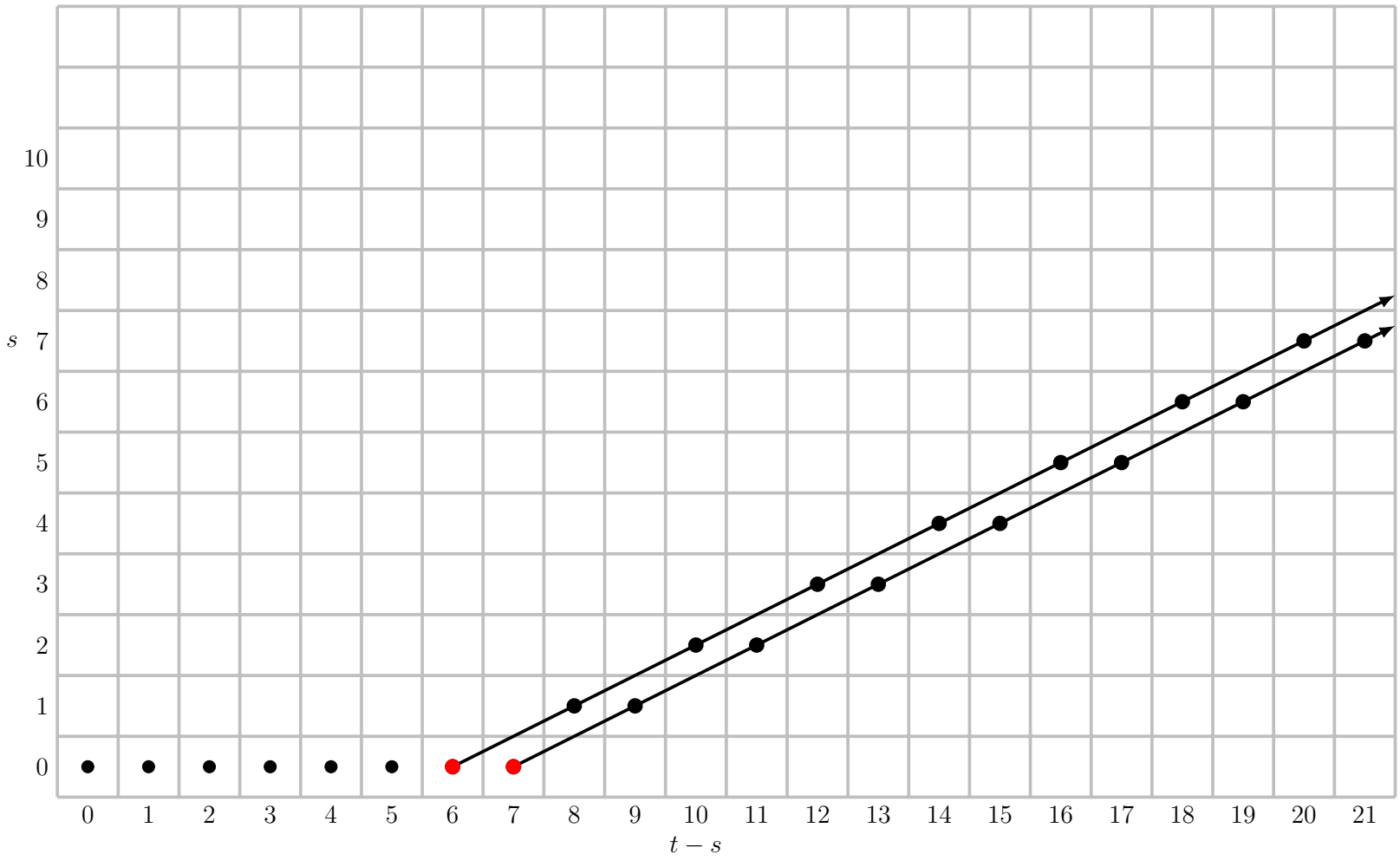
HOPE COLLEGE

$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(8))) \Rightarrow \mathrm{k}(1)_{t-s}(B(8))$$



Adams S.S. for $k(1)_*B(*)$

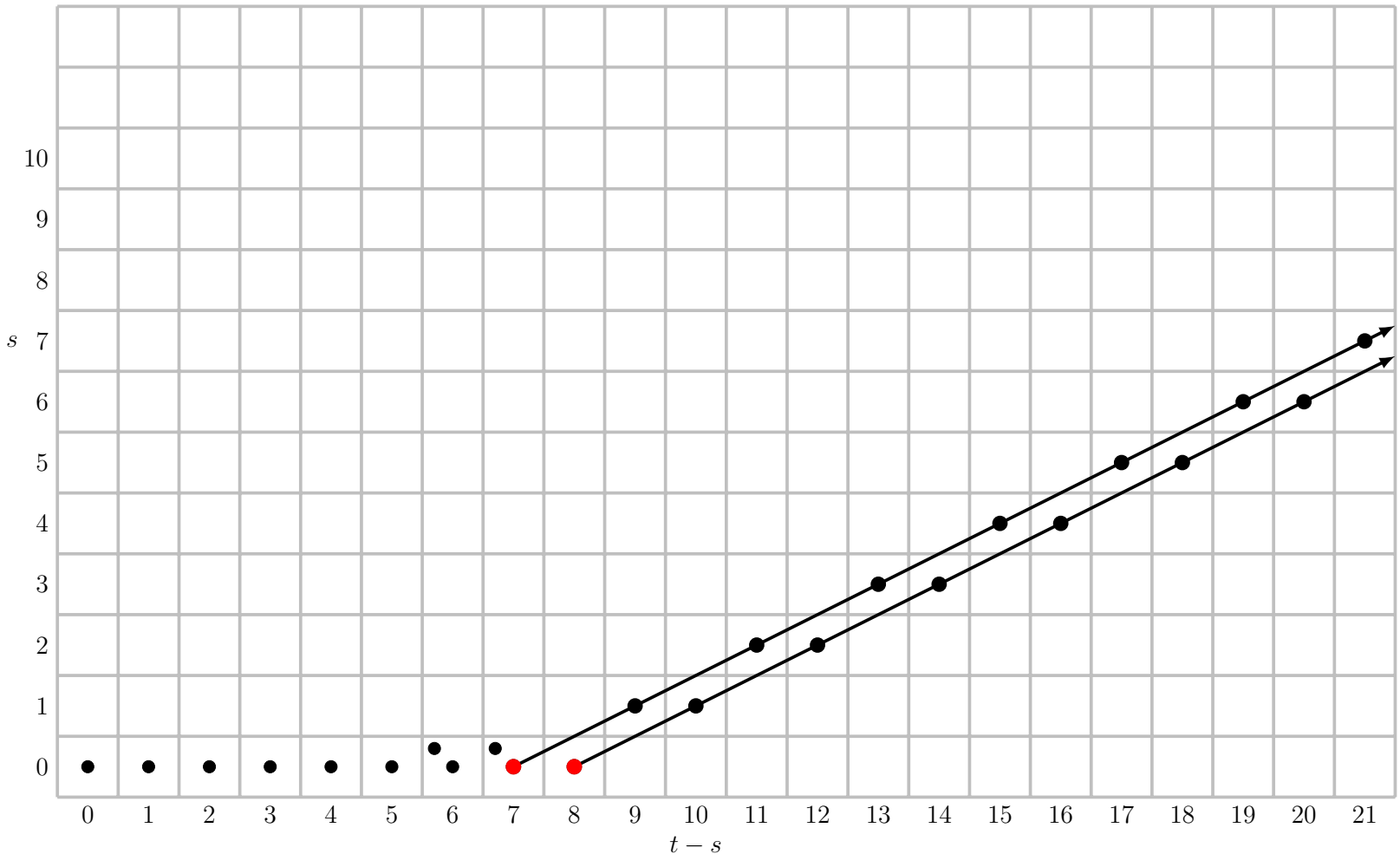
$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(10))) \Rightarrow \mathrm{k}(1)_{t-s}(B(10))$$



Adams S.S. for $k(1)_*B(*)$

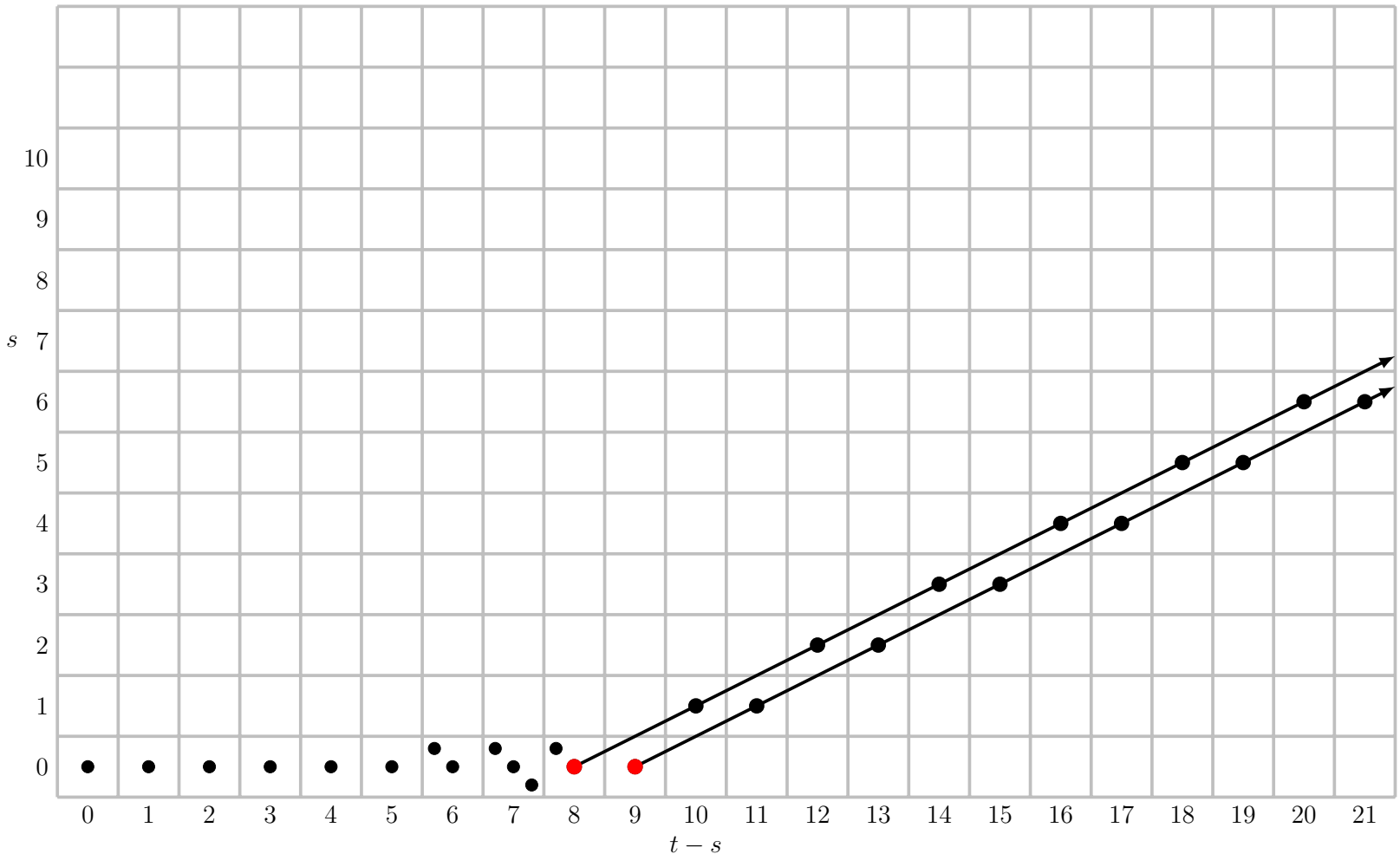
HOPE COLLEGE

$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(12))) \Rightarrow \mathrm{k}(1)_{t-s}(B(12))$$



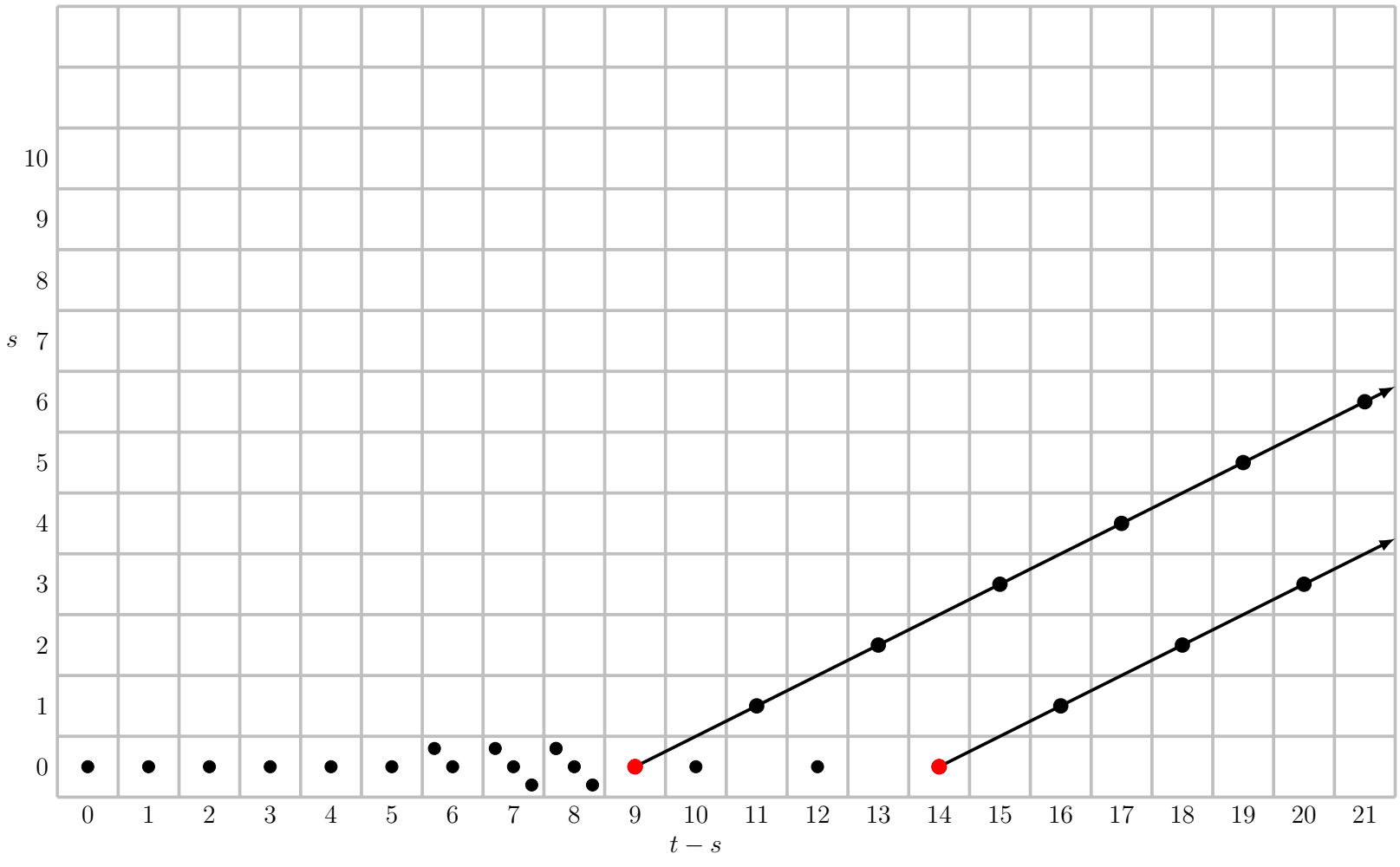
Adams S.S. for $k(1)_*B(*)$

$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(14))) \Rightarrow \mathrm{k}(1)_{t-s}(B(14))$$



Adams S.S. for $k(1)_*B(*)$

$$\mathrm{Ext}^{s,t}_{E(1)}(F_2, H_*(B(16))) \Rightarrow \mathrm{k}(1)_{t-s}(B(16))$$



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Results

HOPE COLLEGE

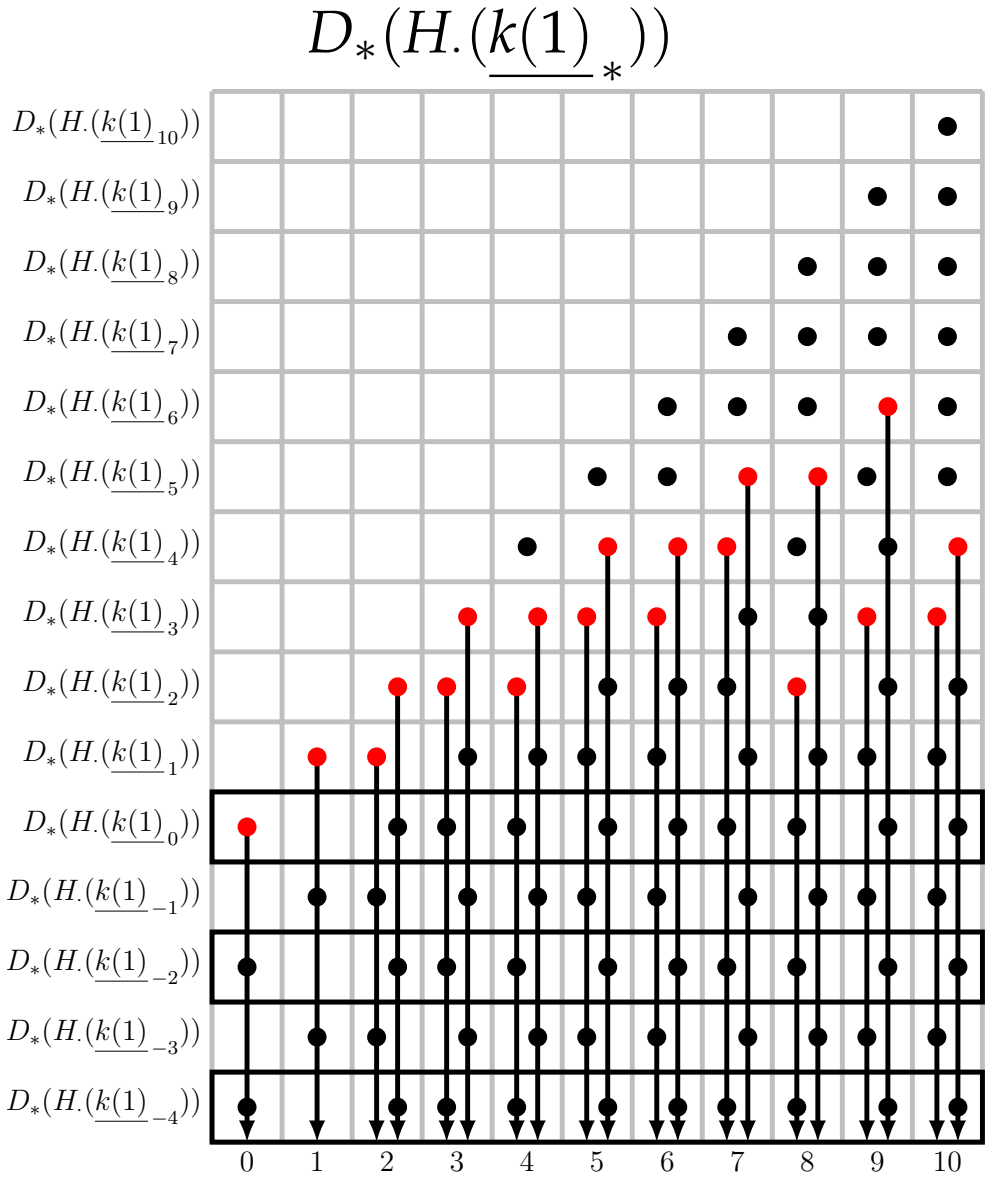
Theorem (Pearson)

The Hopf ring $H_*(\underline{k(r)}_*)$ *is generated by*

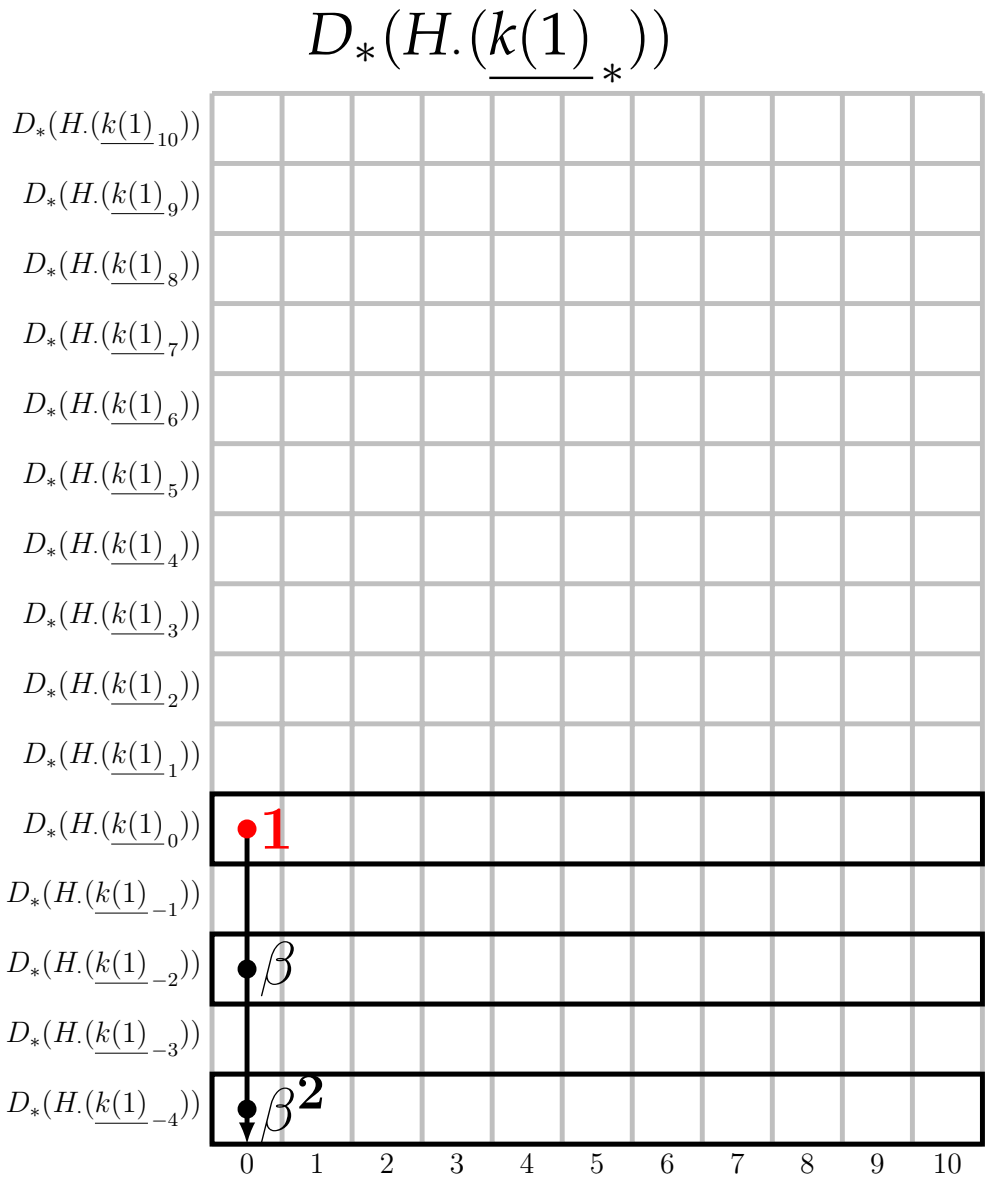
- *stable generators* $e^{-\infty}(H_*(k(r)))$, *and*
- *the homotopy generator* $[v_r] \in H_0(\underline{k(r)}_{2-2^{r+1}})$.

- Many of the relations in this Hopf ring can be read directly from the Ext charts $\mathrm{Ext}^{*,*}_{E(Q_r)}(\mathbb{F}_2, H_*(B(2n)))$
- Future work: it would be nice to have a presentation of this Hopf ring in terms of generators and relations.

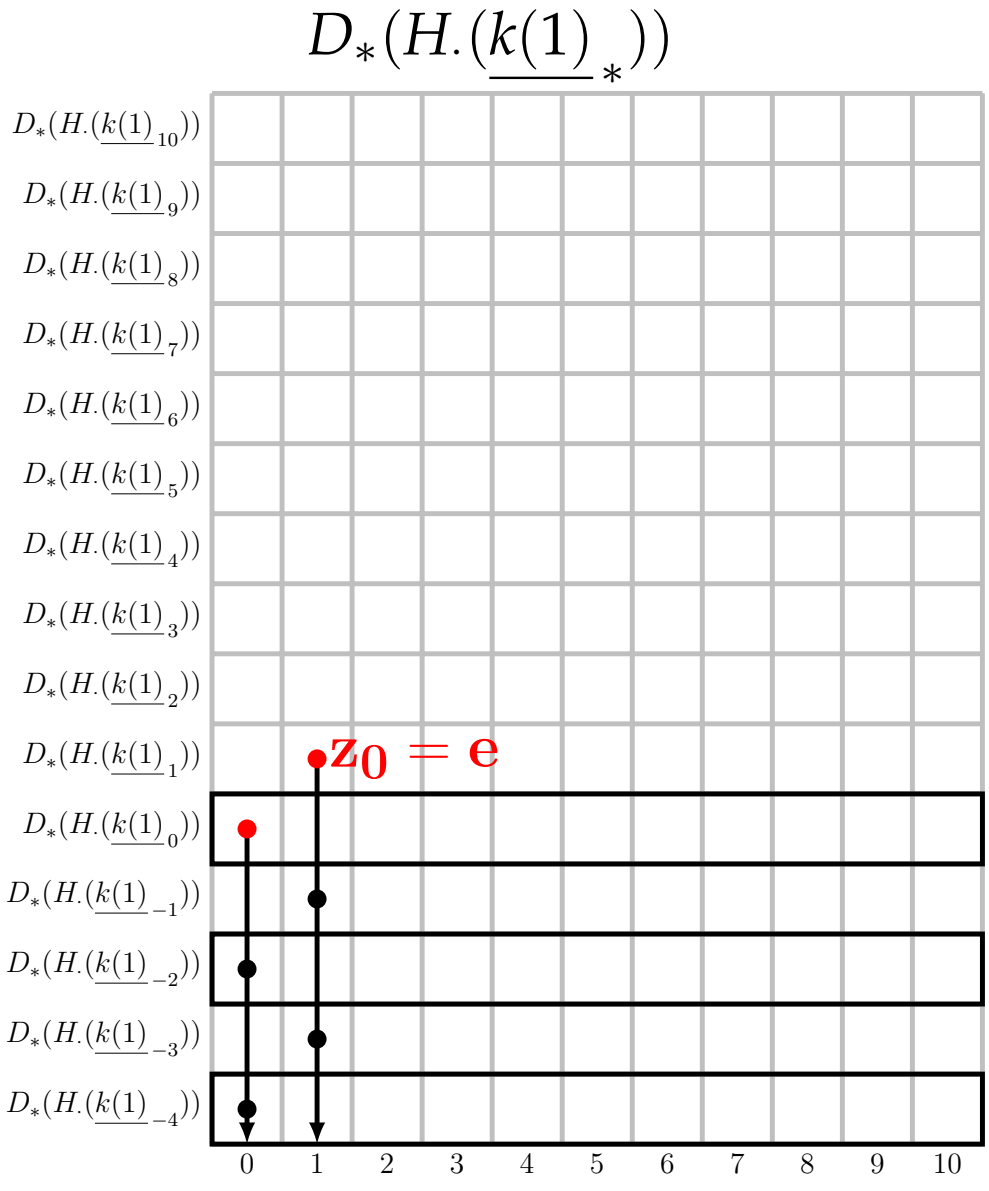
Mod 2 Dieudonné ring for $k(1)$



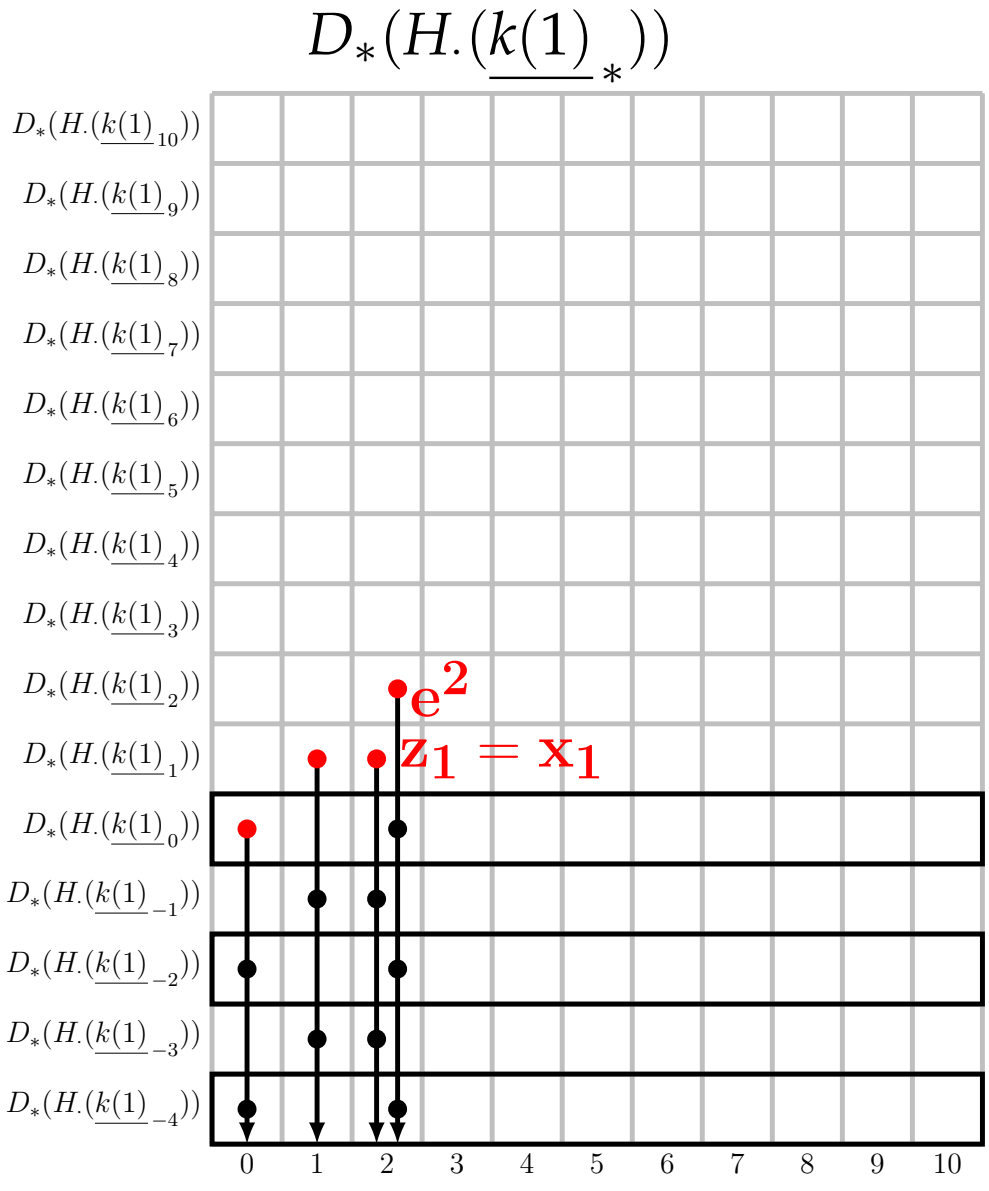
Mod 2 Dieudonné ring for $k(1)$



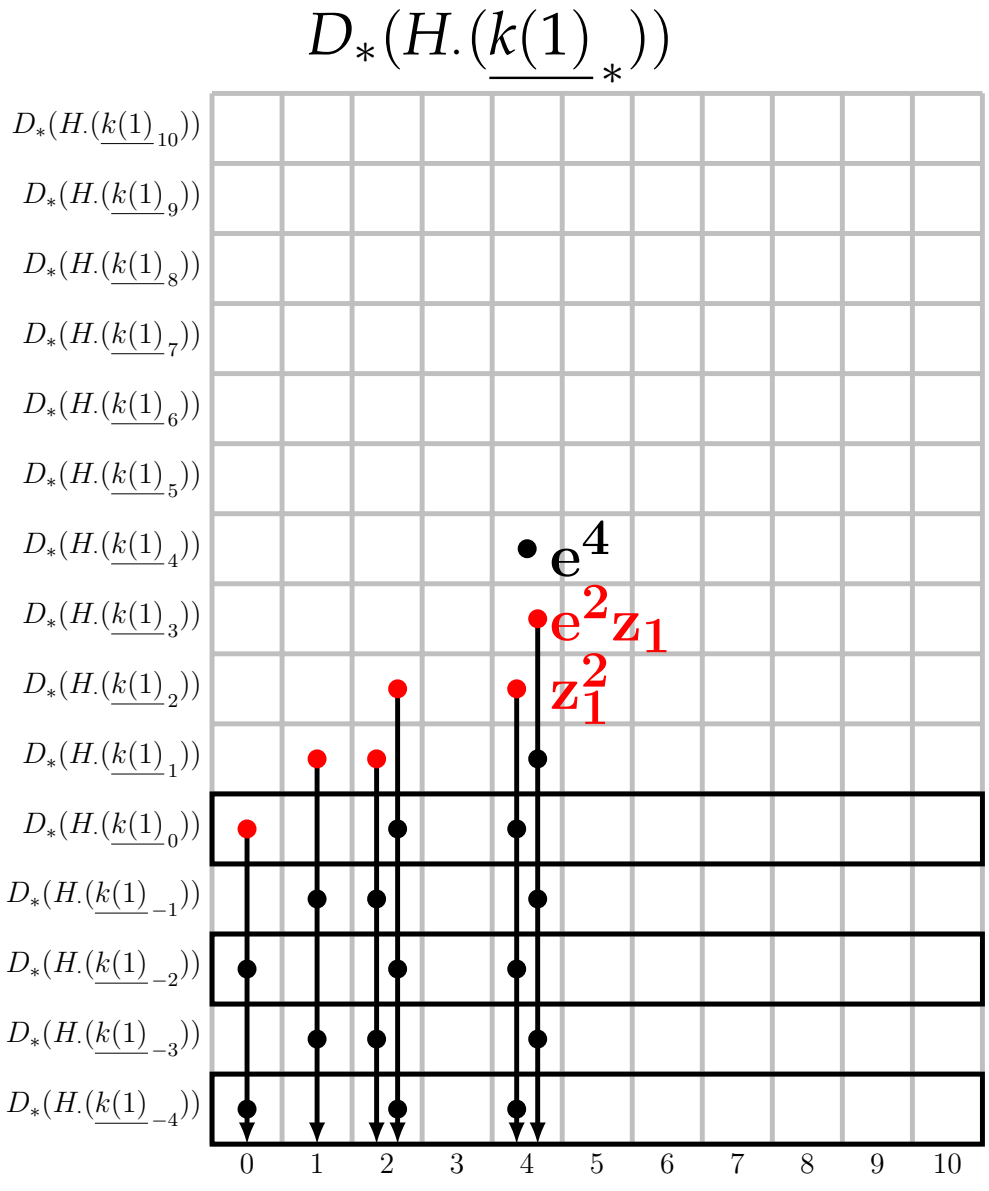
Mod 2 Dieudonné ring for $k(1)$



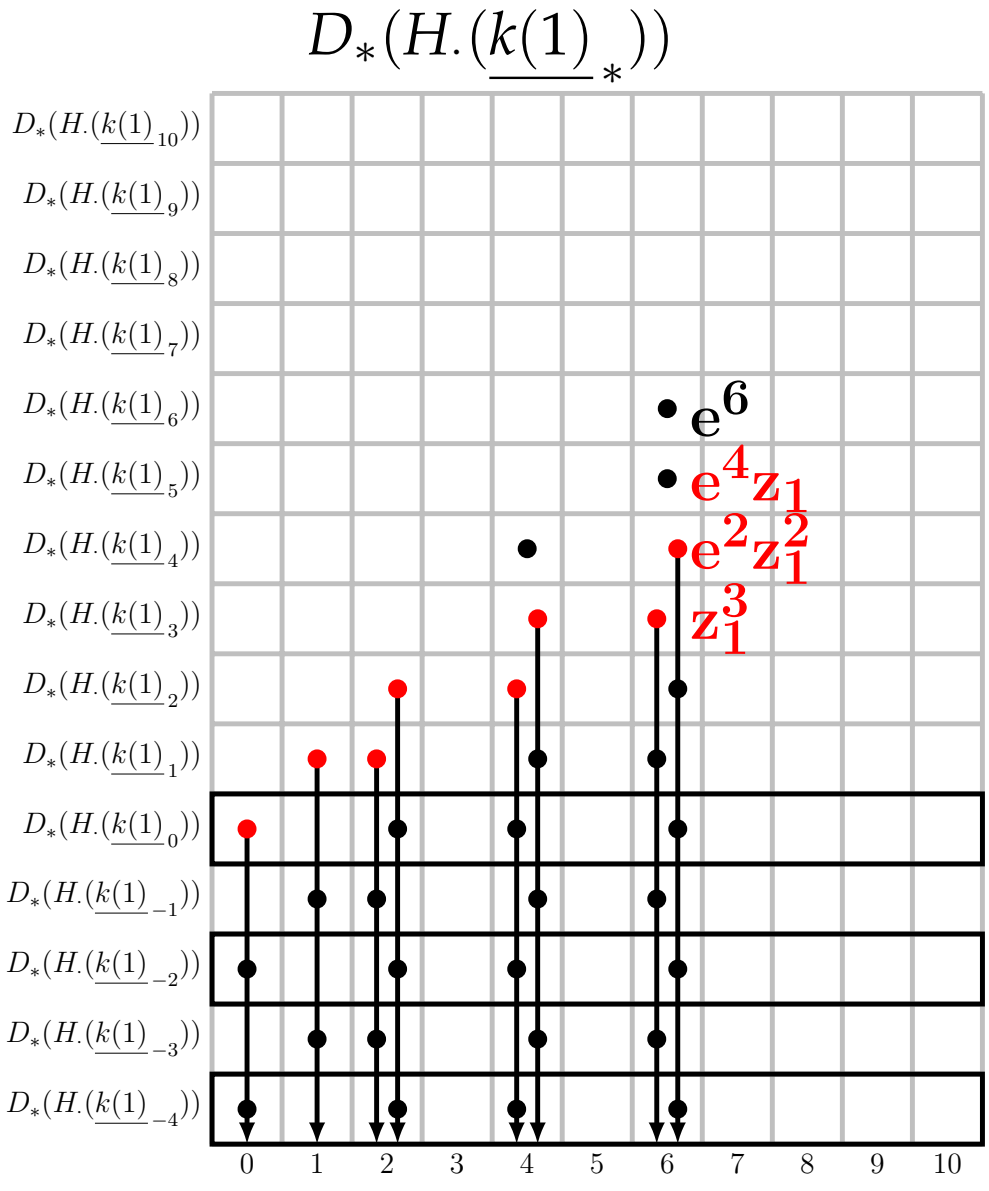
Mod 2 Dieudonné ring for $k(1)$



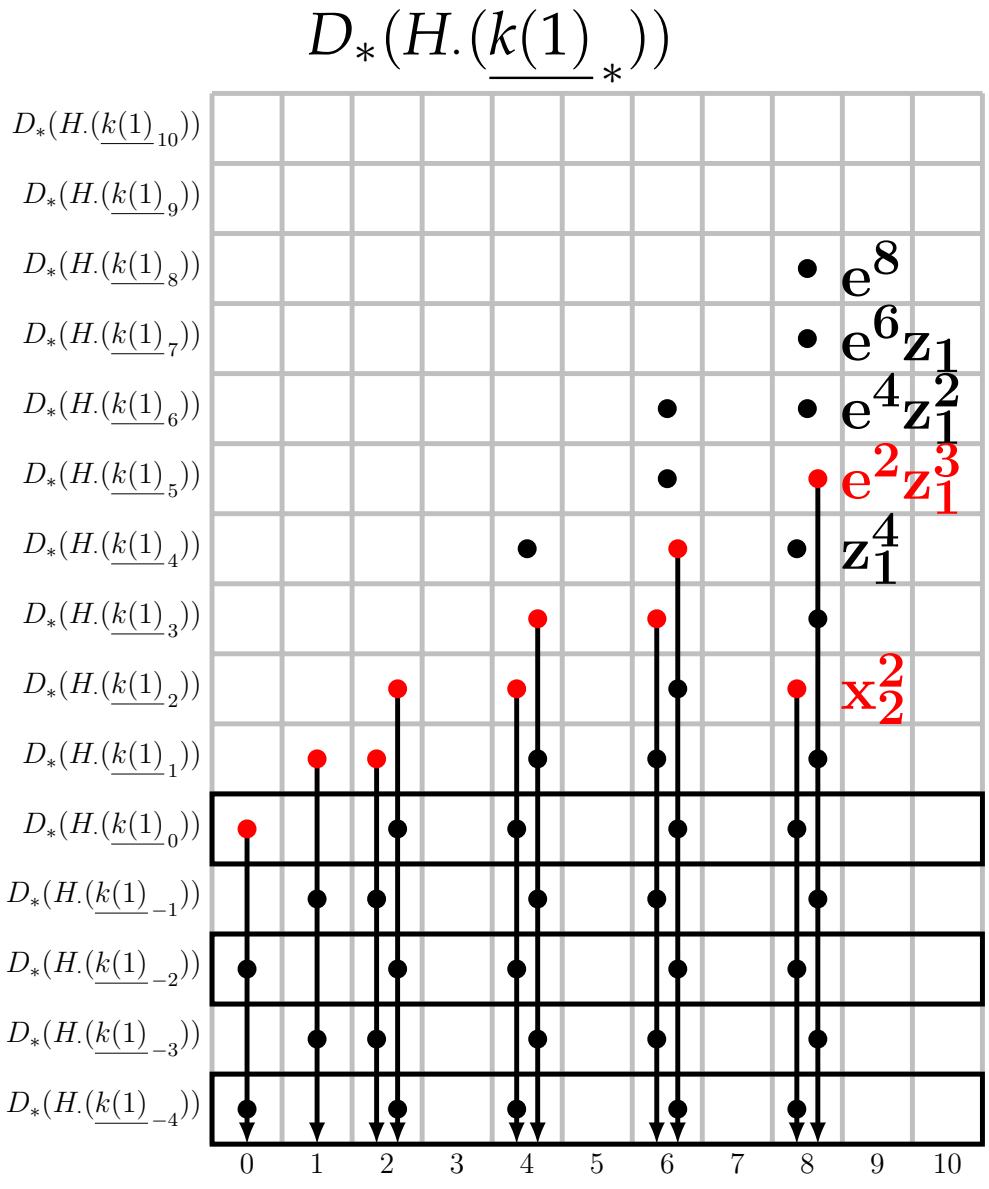
Mod 2 Dieudonné ring for $k(1)$



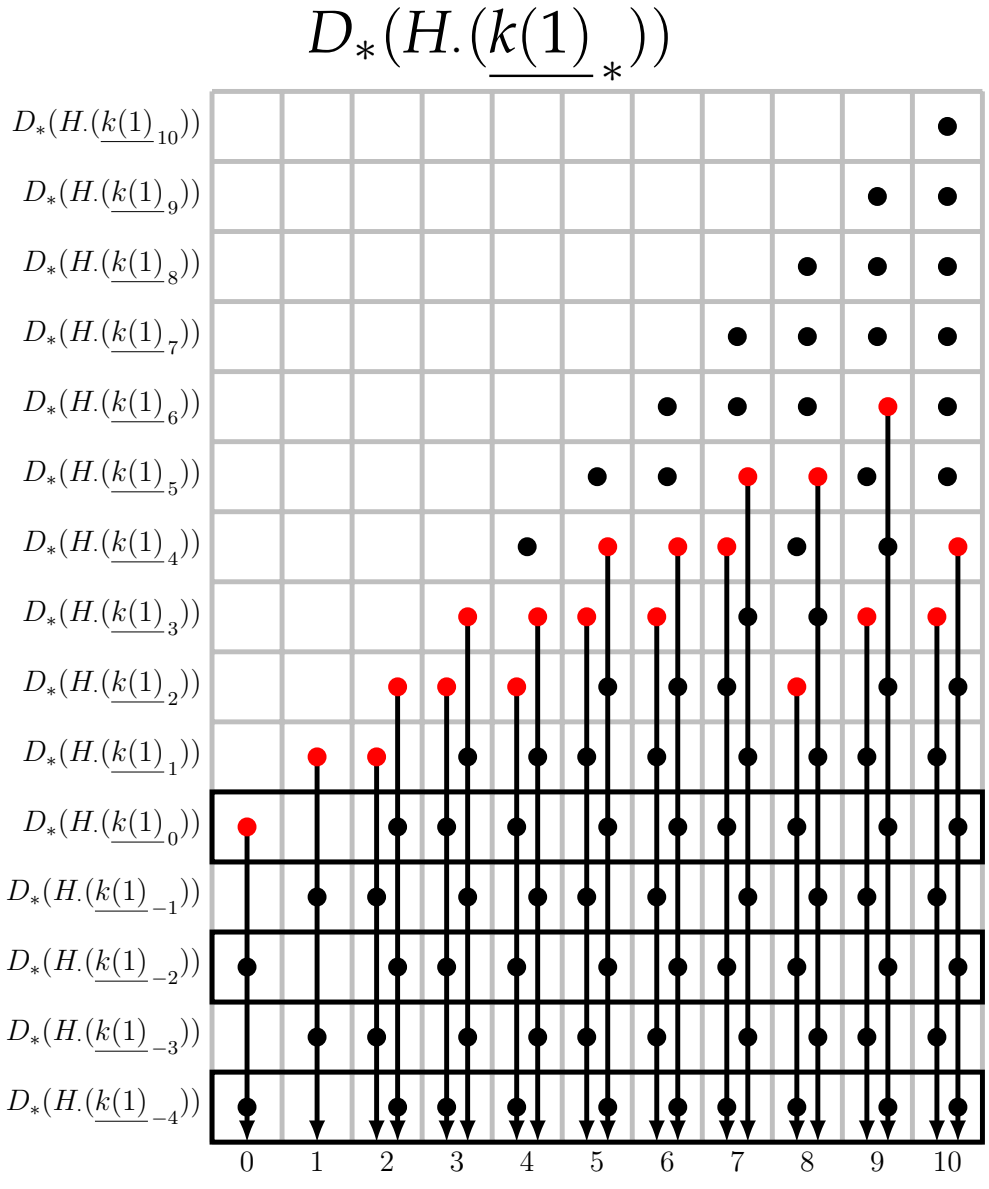
Mod 2 Dieudonné ring for $k(1)$



Mod 2 Dieudonné ring for $k(1)$



Mod 2 Dieudonné ring for $k(1)$



Mod 2 Dieudonné ring for $k(1)$

HOPE COLLEGE

Warning: Margolis homology classes are equivalence classes, and only one element in a Margolis homology equivalence class will correspond to a stable class of the same name in the Hopf ring. For instance, ζ_2^2 is homologous to $\zeta_1^6 + \zeta_2^2 = \xi_2^2$ in $H(H_*(B(8)), Q_1)$, and the destabilization of ξ_2^2 corresponds to the element $x_2^2 \in H_8(\underline{k(1)}_2)$ of the correct bidegree. When the destabilization function is used, there is no ambiguity in naming.

Outline

Key ideas

Things well known

Things less well known

Hopf rings

Dieudonné rings

Brown-Gitler Spectra

Adams spectral sequence

Adams spectral sequence for ko

Dieudonné ring for ko

Margolis homology

Ext for $k(r)$

Dieudonné and Hopf ring results for $k(r)$

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