

GEMS 100 – Section Summaries

1: Functions

Functions are useful in modelling physical, financial, and even sociological situations in addition to being interesting to study from a mathematical perspective. They can be represented symbolically (as a formula), graphically, numerically (as a table of numbers), and verbally. Specifically, a **function** is a rule in which each input is assigned at most one output.

Two important words associated with functions are domain and range. The **domain** of a function is the set of *valid* inputs. The **range** is the set of *actual* outputs.

The notation $f(x)$, pronounced “ f of x ,” is commonly used to denote the output of a function. The letter f is typically used to remind us that this object is a function. The letter x is typically used to denote the input. The f is the *label* representing the function (like your name is the label representing you), and the x is a *placeholder* representing the input.

The **vertical line test** can be used to determine if a graph represents a function. If a vertical line can cross the graph at more than one point, then the graph does not represent a function.

Functions that have different rules for different parts of their domain are called **piecewise functions**. When evaluating a piecewise function for a given input, it is important to make sure you are using the rule defined for the part of the domain containing your input.

2: Graphical Representations of Functions

Graphs are very useful in understanding functions because looking at the graph often gives you a better understanding of the function than just looking at the symbols.

A graph of a function is said to be **increasing** over a particular interval if, for each point in that interval, every point to the right has a greater output. A graph is **decreasing** over an interval if, for each point in that interval, every point to the right has a smaller output.

Although increasing and decreasing describe how the output of the function is changing, the concavity of a function tells how the *rate* of increase or decrease is changing. A graph is **concave up** on intervals where the rate of change increases. A graph is **concave down** on intervals where the rate of change decreases.

Some important points on a graph are the x - and y -intercepts. The **y -intercept** is the value of the function where the graph crosses the y -axis. This is the point where the input (or x -coordinate) of the function is zero. An **x -intercept** is the x -coordinate of a point where the function crosses the x -axis. It is also called a *root* or *zero* of the function. These are the places where the output (or y -coordinate) of the function is zero.

Graphing functions using technology can often be accomplished with a few keystrokes. Sometimes, however, we have to make sure the graph given by a calculator accurately reflects the real graph of the function. Graphing calculators plot points by darkening pixels on the screen. The pixels are then connected by short lines. By this method, a calculator may connect a graph that should not be connected.

When looking at a function on a calculator, choosing a proper viewing window is important. The standard viewing window will not always be the best. In fact, you may not even see your graph in that window.

When calculators are used to find things such as zeros and points of intersection, the numbers given are often not exact. It is important to understand when these numbers are exact and when they are not.

All graphs are not functions. One such graph that may not be a function is that of a scatterplot. A **scatterplot** is a graph used to plot data consisting of two variables.

3: Applications of Graphs

Adding, subtracting, or multiplying a number to the output of a function results in vertical changes in the graph. Adding, subtracting, or multiplying a number to the input of a function results in horizontal changes in the graph. In both cases, adding results in shifts and multiplying results in stretches or compressions. Input changes also tend to be backwards of what you might guess. For example, adding a positive number to the input shifts the graph to the left. A summary of these transformations is given below.

- If $y = f(x)$ is a function and c is a positive number, then
 - The graph of $y = f(x) + c$ is the graph of $f(x)$ shifted up c units.
 - The graph of $y = f(x) - c$ is the graph of $f(x)$ shifted down c units.
- The graph of $y = c \cdot f(x)$ is the graph of $y = f(x)$ vertically stretched by a factor of c for $c > 0$. (If $0 < c < 1$, then the graph of $y = c \cdot f(x)$ can also be referred to as being compressed by a factor of $1/c$.)
- If $y = f(x)$ is a function and c is a positive number then
 - The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left c units.
 - The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted to the right c units.
- The graph of $y = f(cx)$ is the graph of $y = f(x)$ horizontally compressed by a factor of c for $c > 0$. (If $0 < c < 1$, then the graph of $y = f(cx)$ can also be referred to as being stretched by a factor of $1/c$.)

4: Displaying Data

Data involving one variable can be organized in a **frequency distribution** and displayed graphically in a **histogram**. In a histogram, the values are divided into equal intervals on the horizontal axis and the frequencies for the intervals are shown on the vertical axis.

Data involving two variables can be organized in a table and displayed graphically in a **scatterplot**. The independent (or input) variable is shown on the horizontal axis and the dependent (or output) variable is shown on the vertical axis in a scatterplot. Two variables have a **positive association** when an *increase* in the value of one variable leads to an *increase* in the value of the other variable. Two variables have a **negative association** when an *increase* in the value of one variable leads to a *decrease* in the value of the other variable. An **xy-line** or **trend line** is a scatterplot that represents a function where the consecutive points are connected with line segments.

5: Describing Data

The two most common ways of measuring the center (or average) of a data set are the mean and median. To determine the **mean** of a data set, you add up the data values and divide by the number of pieces of data. The **median** is the middle number of an *ordered* set of data. If you are trying to find the median for an even number of values, then the median is the mean of the middle two in the ordered set.

If a histogram of a data set close to symmetric, the mean and median will be approximately the same. A histogram is *skewed to the right* if the righthand ‘tail’ is longer than the lefthand ‘tail.’ The mean will be larger than the median when a distribution is skewed to the right. The opposite is true of a histogram skewed to the left.

The simplest measure of spread is range. The **range** of a set of numbers is the smallest number subtracted from the largest number. Other measures of spread include variance and standard deviation. The **variance** of a set of numbers is the mean of the squared differences between each number of the set and the mean of the set. **Standard deviation** is the square root of the variance. Standard deviation can be thought of roughly as the average distance between the individual data points and the mean of the data set. The larger the standard deviation, the greater the spread of the data.

Percentiles are frequently used to describe standardized test scores such as the ACT or the SAT. The **n th percentile** of a distribution is the number such that n percent of the observations fall at or below it. The median is always the 50th percentile. Other commonly reported percentiles are the 25th and the 75th. These percentiles are also called quartiles. The 25th percentile is known as the first quartile and the 75th percentile is known as the third quartile. (The median is the 50th percentile or the second quartile.) Just as the median divides a distribution into two groups with the same number of data points in each group, the 1st, 2nd, and 3rd quartiles divide a distribution into four groups with the same number of data points in each group. The difference between the 3rd quartile and the 1st quartile is called the **interquartile range**. This is also a measure of the spread of the data.

6: Multivariable Functions

A **multivariable function** is a function that has more than one input. We focused on multivariable functions that had two inputs. These types of functions can be graphed on a computer. By fixing the output, these types of functions can be graphed in a plane as a **contour curve**. Collections of these contour curves are often seen in topographical and weather maps. A contour curve on a weather map is called an **isotherm**.

By fixing one of the inputs in a multivariable function a single variable function can be made. These are often useful in understanding the relationship between an input variable and output variable of the function.

7: Linear Functions

A **linear function** is one in which the rate of change is constant. The **rate of change** is the change in output divided by the change in input. The **y -intercept** is where the graph crosses the y -axis. The **slope** is the rate of change, or (change in output)/(change in input). Symbolically, $\text{slope} = (y_2 - y_1)/(x_2 - x_1)$, where (x_1, y_1) and (x_2, y_2) are any two points on the line. A linear function with a positive slope is an increasing function and linear function with a negative slope is a decreasing function.

The most common way to write a linear equation is using the **slope-intercept form**, $y = mx + b$, where y is the output, m is the slope, x is the input, and b is the y -intercept.

You can find the equation of a line by finding the coordinates of any two points on the line, (x_1, y_1) and (x_2, y_2) . Once you have the two points, you can use the two points to find the slope, substitute the value of the slope for m and the coordinates of one of your points for x and y into the equation $y = mx + b$, solve for b , and then write your equation in slope-intercept form.