FIFTY-SEVENTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

Part I

Tuesday October 1, 2013

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(E) none of the others." If you believe none of the first four alternatives is correct, choose response (E).

6. Any scientific or graphing calculator is permitted on Part I. (Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.) No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may now open the test booklet and begin.

1. How many three-digit numbers are perfect cubes?

   A: 5 B: 6 C: 7 D: 8 E: 9

2. What is $1 - 2 + 3 - 4 + 5 - 6 + \cdots + 2013$?

   A: -1007 B: -1006 C: 0 D: 1006 E: 1007

3. A sequence of positive integers is formed by the following process. The first term is 2013. Each term after the first is the sum of the squares of the digits of the previous term. What is the seventh term?

   A: 85 B: 89 C: 99 D: 101 E: 145

4. Find the area of the polygon whose vertices are $(0, 0), (2, 0), (3, 1), (1, 2), (0, 1)$.

   A: 3 B: 4 C: 5 D: 6 E: none of the others

5. All six sides of a hexagon $ABCDEF$ have length 1. Angles A and D are right angles, and the other four angles are $135°$. What is the area of the hexagon?

   A: $2 - \sqrt{2}$ B: 2 C: $1 + \sqrt{2}$ D: $\frac{3}{2}\sqrt{3}$ E: $1 + 2\sqrt{2}$

6. I am driving in the right lane of a superhighway at a constant speed of 70 mph. I note a car behind me, traveling in the same direction in the left lane, at a higher speed. When this car passes me, I start counting time. After 10 seconds have elapsed, I note the position of the faster car. I continue counting, and discover that I reach the noted location in 2 seconds. How fast is the other car traveling? (Assume that both cars are traveling at constant speeds, and that times have been counted accurately.)

   A: 60 mph B: 75 mph C: 84 mph D: 90 mph E: none of the others

7. Suppose $10x + 90y + 2z = 100$ and $45x + 5y - z = 50$. Find $x + y$.

   A: 11 B: 0 C: 1 D: 2 E: none of the others

8. A regular n-gon is inscribed in a circle of radius r. What is the smallest value of n for which the length of each side of the n-gon is strictly less than r?

   A: r B: $r + 1$ C: 6 D: 7 E: none of the others

9. The vertices of an equilateral triangle lie on a circle. What proportion of the area lying within the circle also lies inside the triangle?

   A: $\frac{1}{2}$ B: $\frac{3\sqrt{3}}{4\pi}$ C: $\frac{\sqrt{3}}{\pi}$ D: $\frac{3\sqrt{3}}{2\pi}$ E: none of the others

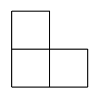
10. Pollux lists the integers from 1 to 100. He writes odd digits in blue and even digits in red. (So, when he writes 70, the 7 is blue and the 0 is red.) How many more blue digits are there than red digits?

A: 1 B: 5 C: 6 D: 10 E: 11

11. Angela's current age in years is exactly one third of her father's. Four years ago she was exactly one fourth of his age. How many years from now will she be just half his age?

A: 8 years B: 2 years C: 4 years D: 6 years E: 12 years

12. What are the dimensions of the smallest square that can be tiled (covered with no excess and no overlapping) by blocks composed of three 1×1 squares arranged in the following shape?



A: 3×3 B: 4×4 C: 5×5 D: 6×6 E: 12×12

13. Which of these describes the solution set of the following system of equations?

$$\begin{aligned} 1x + 2y + 3z &= 4 \\ 5x + 6y + 7z &= 8 \\ 9x + 10y + 11z &= 12 \end{aligned}$$

A: no solution B: one point C: two points D: one line
E: two lines

14. Alex, Beatrix, and Clarix each draws a regular polygon. Alex's polygon has 12 sides, Beatrix's has 18 sides, and Clarix's has n sides. If each angle of Beatrix's polygon is the arithmetic mean of an angle from Alex's polygon and an angle from Clarix's polygon, what is n?

A: 20 B: 24 C: 30 D:32 E: 36

15. From a standard pack of 52 playing cards, seven cards are dealt face-down in front of you. You are told that two of the cards are red on the face-down side and five are black on the face-down side. You choose two cards at random and turn them face up. Let A be the probability that both your cards are red, let B be the probability that exactly one of your cards is red, and let C be the probability that neither of your cards is red. Then

A: $A < B < C$ B: $A < B = C$ C: $A < C < B$ D: $A > B > C$
E: none of the others

16. July 4, 2013, was a Thursday. On what day of the week will July 4, 2049, fall?

A: Wednesday B: Thursday C: Friday D: Saturday E: Sunday

17. The sum of the coefficients in the expansion of $(2 - x^2)^{2013}$ is

A: -2^{2013} B: -1 C: 1 D: 2^{2013} E: none of the others

18. Suppose that $\log_{20} \log_{13} \log_{10} \log_2 x = y$. Then what is x?

A: 45^y B: 5200^y C: $20^{13^{10^{2^y}}}$ D: $2^{10^{13^{20^y}}}$ E: none of the others

19. There are 150 numbers divided into two groups. The average of the first group is 82, the average of the second group is 85, and the overall average is 83. How many numbers are there in the first group?

A: 80 B: 50 C: 100 D: 65 E: none of the others

20. Suppose that s and t are the roots of $ax^2 + bx + c$ where a and c are nonzero. What can you say about the roots of $cx^2 + bx + a$?

A: $1/s$ and $1/t$ B: s and t C: s/t and t/s D: $s - t$ and $t - s$
E: none of the others

21. A sequence is defined by the following rules: $a_0 = 1$, $a_1 = 2$, and, for $n \geq 2$, a_n is the remainder when $a_{n-2} \cdot a_{n-1}$ is divided by 5. Which of the following is equal to a_{1000}?

A: 0 B: 1 C: 2 D: 3 E: 4

22. Some cards have been prepared with a letter of the alphabet on one side and a numeral $(0, 1, \ldots, 9)$ on the other side. For example, a card might have a Q on one side and a 5 on the other side. You are shown four cards which display:

8	M	9	N

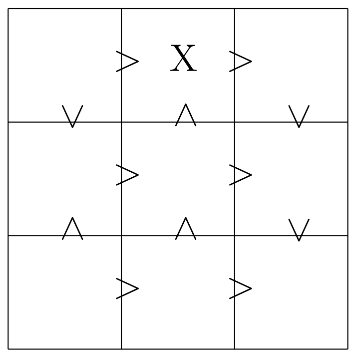
It is further proposed that these four cards have the additional property that if the letter on one side is M, then the number on the other side is 9. Which cards must be turned over in order to determine whether these four cards have the proposed property? (Don't turn over more cards than necessary.) The cards that need to be turned are

A: M B: M and 8 C: M, 8, and 9 D: M, N, 8, and 9
E: none of the others

23. Let A be the product of the first 2013 prime numbers, and let B be the product of the next 2013 prime numbers. What is the units digit of $A + 5B$?

A: 0 B: 1 C: 3 D: 5 E: none of the others

24. The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged in the 9 small squares in the figure below so that each square contains a different number. The numbers obey the inequalities indicated on the boundaries between adjacent squares. What number must located in the square marked "X"?



A: 3 B: 4 C: 6 D: 7 E: none of the others

25. A regular 12-gon is inscribed in a circle. How many line segments are there that connect two vertices of the polygon but that are neither edges of the polygon nor diameters of the circle?

A: 48 B: 54 C: 72 D: 96 E: 108

26. For x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the graphs of $f(x) = x^{100}$ and $g(x) = \tan x$ intersect in how many points?

A: 1 point B: 2 points C: 3 points D: an infinite number of points E: none of the others

27. Which of the following numbers is closest to $\log_2 \left(2^{999} + 2^{1000}\right)$?

A: 999 B: 1000 C: 1005 D: 1001 E: 2000

28. The sides of a rhombus are 5 units long. At least one of the diagonals is a whole number of units long. How many different rhombi fit this description?

A: None B: 7 C: 8 D: 9 E: 10

29. Say that a standard parabola in the xy-plane is one whose points satisfy an equation in one of the following forms.

$$y = Ax^2 + Bx + C$$

$$x = Ay^2 + By + C$$

How many points can be in the intersection of two distinct standard parabolas?

A: 2, 4 B: 0, 2, 4 C: 0, 1, 2, 4 D: 0, 1, 2, 3, 4 E: none of the others

30. The outside surface of a large cubical block is painted and then broken into unit cubes. If exactly 294 of the unit cubes were painted on just one side, what are the dimensions of the large cube?

A: $6 \times 6 \times 6$ B: $7 \times 7 \times 7$ C: $8 \times 8 \times 8$ D: $9 \times 9 \times 9$
E: none of the others

31. Suppose $x + y = a$ and $x^3 + y^3 = b$ where $a \neq 0$. Find xy in terms of a and b.

A: $\frac{a^3-b}{3a}$ B: $\frac{a^3+b}{3a}$ C: $\frac{a^3-b}{3b}$ D: $\frac{a^3+b}{3b}$ E: none of the others

32. What is the last decimal digit of $2013^{2013^{2013^{2013}}}$?

A: 1 B: 3 C: 5 D: 7 E: 9

33. When the number n is expressed in base b, it is written 2013. (Here b is an integer greater than 3.) Which of the following statements about n is necessarily true?

A: n is even B: n is odd C: n is a multiple of 3
D: n is a multiple of 11 E: n is not a multiple of 5

34. If a, b, and c are integers that satisfy $abc + a + b + c = ab + bc + ca + 2013$, then how many possible values are there for b?

A: 4 B: 6 C: 8 D: 12 E: 18

35. The expression $x^2 - y^2 + z^2 - 2xz + x + y - z$ has a factor

A: $x - y - z + 1$ B: $x - y + z + 1$ C: $x + y - z + 1$
D: $-x + y - z + 1$ E: $-x - y + z + 1$

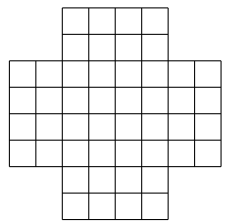
36. Of the 26 numbers $144, 145, 146, \ldots, 168, 169$, twenty-five of them are arranged into a 5×5 array so that the sums of all rows and the sums of all columns are the same. Which could be the number that was omitted?

A: 151 B: 152 C: 153 D: 154 E: 155

37. Three women and three men attend a party. Each party guest is given either a red hat or a green hat. There are three hats of each color, and the hats are distributed randomly. What is the probability that all of the women will have the same color hat?

A: 1/20 B: 1/10 C: 1/8 D: 1/2 E: none of the others

38. A 2×2 square is removed from each corner of a chessboard, leaving a cross shape with 48 squares:



Let n be the number of ways to color 8 squares red in such a way that each row and each column has exactly one red square. Which of the following is closest to n? (Distinct colorings that would be the same after rotation or reflection should be counted separately.)

A: 100 B: 300 C: 600 D: 1000 E: 1500

39. There are three decks of 18 cards each. One deck has cards numbered 1 to 9, colored red and orange, with one of each number/color combination. The second deck has cards numbered 1 to 6, colored red, orange, and yellow, with one of each combination. The third deck has cards numbered 1 to 3, colored red, orange, yellow, green, blue, and purple, with one of each combination. Kendrix knows all of this. Kendrix chooses one of the three decks at random without knowing which it is. She shuffles the deck and then looks at the top card. What is the probability that the card viewed is sufficient to determine which deck Kendrix has?

A: 0 B: 1/18 C: 1/9 D: 1/6 E: 1/3

40. How many nine-digit numbers (without leading zeros) are palindromes (giving the same number when the digits are read backward) and are also divisible by 9?

A: 6561 B: 9000 C: 10000 D: 43020 E: 59049

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.