MMPC STUDENT CODE

FIFTY-SIXTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

December 12, 2012

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the blank pages at the end of the booklet (pages 7, 8, 9) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. A permutation on $\{1, 2, \ldots, n\}$ is an ordered arrangement of the numbers. For example, 32154 is a permutation of $\{1, 2, 3, 4, 5\}$. Does there exist a permutation $a_1 a_2 \ldots a_n$ of $\{1, 2, \ldots, n\}$ such that $i + a_i$ is a perfect square for every $1 \le i \le n$ when

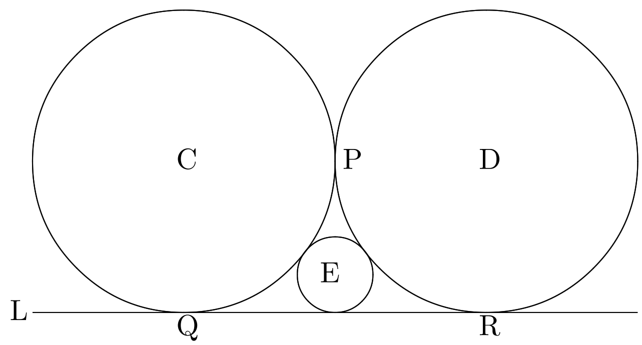
   a) $n = 6$?

   b) $n = 13$?

   c) $n = 86$?

   Justify your answers.

2. Circle C and circle D are tangent at point P. Line L is tangent to C at point Q and to D at point R where Q and R are distinct from P. Circle E is tangent to C, D, and L, and lies inside triangle PQR. C and D both have radius 8. Find the radius of E, and justify your answer.



3. (a) Prove that $\sin 3x = 4\cos^2 x \sin x - \sin x$ for all real x.

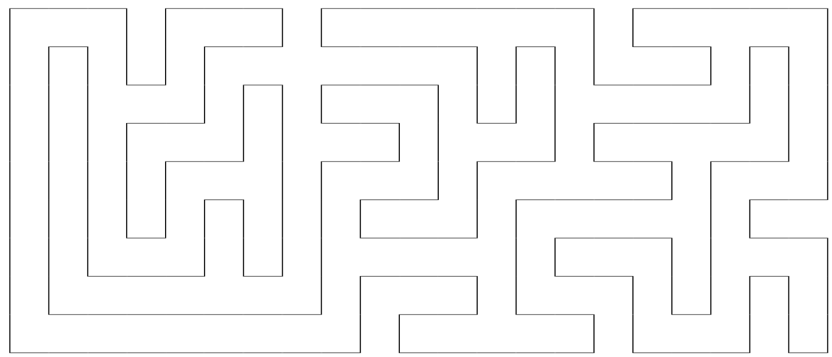
   (b) Prove that

$$(4\cos^2 9^\circ - 1)(4\cos^2 27^\circ - 1)(4\cos^2 81^\circ - 1)(4\cos^2 243^\circ - 1)$$

   is an integer.

4. Consider a $3 \times 3 \times 3$ stack of small cubes making up a large cube (as with the small cubes in a Rubik's cube). An ant crawls on the surface of the large cube to go from one corner of the large cube to the opposite corner. The ant walks only along the edges of the small cubes and covers exactly nine of these edges. How many different paths can the ant take to reach its goal?

5. Let m and n be positive integers, and consider the rectangular array of points (i, j) with $1 \leq i \leq m$, $1 \leq j \leq n$. For what pairs m, n of positive integers does there exist a polygon for which the mn points (i, j) are its vertices, such that each edge is either horizontal or vertical? The figure below depicts such a polygon with $m = 10$, $n = 22$. Thus 10, 22 is one such pair.



(continued solutions)

The Michigan Mathematics Prize Competition is an activity of the
Michigan Section of the Mathematical Association of America.

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