FIFTY-SIXTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

Tuesday October 2, 2012

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(E) none of the others." If you believe none of the first four alternatives is correct, choose response (E).

6. Any scientific or graphing calculator is permitted on Part I. (Unacceptable machines include computers, PDAs, pocket organizers, cell phones, and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.) No other devices are permitted.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may now open the test booklet and begin.

1. Consider five boxes with the dimensions listed below. Which one has the largest surface area?

   A: $2 \times 2 \times 16$ B: $1 \times 8 \times 8$ C: $1 \times 4 \times 16$ D: $2 \times 4 \times 8$
   E: All have the same surface area

2. $\log_8(\log_2 16) =$

   A: $\frac{2}{3}$ B: $\frac{3}{2}$ C: $\frac{1}{2}$ D: 1 E: none of the others

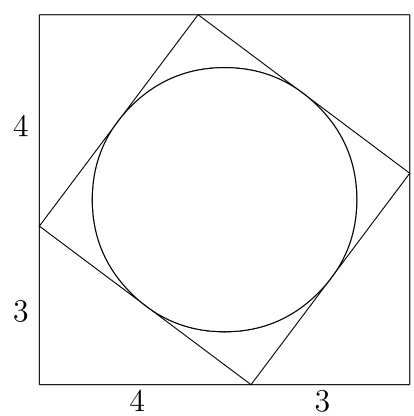
3. In the xy-plane, the set of points that are equidistant from the point $(0, 1)$ and the line $y = 0$ forms

   A: a parabola B: a hyperbola C: a circle D: an ellipse
   E: none of the others

4. What is the equation of the straight line through the point $(2, -3)$ that is parallel to $2x+3y = 1$?

   A: $2x - 3y = 13$ B: $2x + 3y = 5$ C: $-3x + 2y = -15$
   D: $2x + 3y = 0$ E: none of the others

5. The larger square in the figure below has side 7. The corners of the smaller square touch the sides of the larger square, as shown. What is the area of the circle inscribed in the smaller square?



   A: 25π B: $49\pi/4$ C: 12π D: $25\pi/4$ E: none of the others

6. The faces of two dice are numbered in a non-standard manner. One has one 1, two 2s, and three 3s. The other has three 4s, two 5s, and one 6. The dice are carefully balanced so that each face is equally likely to land on top. When the pair of dice are rolled what is the probability that the sum of the two dice is 7?

   A: 7/18 B: 5/18 C: 1/3 D: 1/4 E: none of the others

7. For integer values of n with $0 \leq n \leq 2012$, the minimum value of $(2012-n)!(n!)^2$ occurs when n is equal to

   A: 1006 B: 671 B: 670 D: 45 E: 44

8. If $y = ax$ bisects the area of the triangle in the xy-plane bounded by $x + y = 10,\ 3x - y = 0,$ and the x-axis, what is a?

A: $\dfrac{5}{3}$ B: $\dfrac{3}{5}$ C: $\dfrac{15}{4}$ D: $\dfrac{4}{15}$ E: none of the others

9. The arithmetic mean of 8 and 18 exceeds the geometric mean of these numbers by

A: 3 B: 1 C: 5 D: 2 E: none of the others

10. From a pile of pennies, nickels, dimes, and quarters I grab a handful of coins. The combined value of the fifteen coins is $1.05, and there are 3 more quarters than nickels. How many dimes do I have?

A: 1 B: 2 C: 3 D: There is no solution to this problem. E: none of the others

11. Suppose that $f(a+b) - 2f(a) + f(a-b) = 2f(b-1)$ for all integers a and b. Suppose also that $f(1) = 2$. Then $f(-1) + f(0) =$

A: $\frac{1}{2}$ B: $\frac{1}{4}$ C: $-\frac{1}{2}$ D: $-\frac{1}{4}$ E: none of the others

12. We have two similar solids. The volume of the larger one is $16\sqrt{2}$ times the volume of the smaller one. The sum of their surface areas is 216 square inches. What is the surface area of the smaller solid?

A: 16 square inches B: 24 square inches C: 48 square inches D: 18 square inches
E: none of the others

13. For what values of a does $|x + a| + |x + 2| = 5$ have an infinite number of solutions?

A: -7 and 3 B: 7 only C: -3 and 7 D: -7 only
E: No such value of a exists.

14. A snail is at the bottom of a well, 115 feet deep. On day 1, it starts climbing up the side. That night it rests and slips down one foot. It repeats this process, climbing up 3 feet each day and sliding down one foot each night until it reaches the top of the well. On what day number with the snail reach the top?

A: 39 B: 57 C: 58 C: 115 E: none of the others

15. Two asteroids in space hurtling directly toward each other. Initially, they are 2012 kilometers apart. Asteroid A travels at a rate of 1720 kilometers per hour. Asteroid B is travels at a rate of 1280 kilometers per hour. How many kilometers apart are the two asteroids one minute before they crash?

A: $\dfrac{1720+1280}{60}$ B: $\dfrac{\sqrt{2012^2+3000^2}}{(1720+1280)60}$ C: $\dfrac{1720+1280}{60\sqrt{2012^2+3000^2}}$
D: 2012 E: none of the others

16. $\left(\dfrac{1+i}{\sqrt{2}}\right)^{4,444,444} =$

A: $-\dfrac{1}{2^{2,222,222}}$ B: $\dfrac{1}{2^{2,222,222}}$ C: 1 D: -1
E: none of the others

17. One car goes around a track in 30 seconds, and a second car goes around the same track in 50 seconds. How long will it take the faster car to gain one lap?

    A: 75 seconds B: 150 seconds C: 300 seconds
    D: 1500 seconds E: none of the others

18. The fraction $\frac{1}{4}$ is expanded in base b where b is an integer with $b \geq 2$. The expansions need not terminate. If only the digits 0 and 1 are required, then the possible values of b are

    A: 2, 4 B: 2, 4, 5 C: 3, 4, 5 D: 3, 4, 6, 7
    E: There are no possible values for b.

19. The polynomial $x^3 + ax^2 + bx + c$ has three distinct real roots. Two of them are 1 and 2. Which of the following statements must be true?

    A: $c = 2a + 6$ B: $c = 2a - 6$ C: $a + b + c = 1$ D: a and c are either both positive or both negative E: none of the others

20. How many integers satisfy the inequalities

$$x(x-6)(x-8) \geq 0 \text{ and } \frac{x-5}{x^2-3x+2} \leq 0?$$

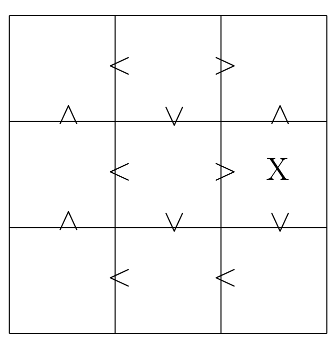
    A: 1 B: 2 C: 3 D: 4 E: none of the others

21. Suppose $a_1 = 2,\ a_2 = 1$ and $a_{n+2} + a_{n+1} + a_n = 7$ for $n \geq 1$.

    Find $\sum_{k=1}^{113} a_k$.

    A: 255 B: 259 C: 262 D: 266 E: none of the others

22. The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged in the 9 small squares in the figure below so that each square contains a different number. The numbers obey the inequalities indicated on the boundaries between adjacent squares. What number must located in the square marked "X"?



    A: 9 B: 8 C: 7 D: 6 E: none of the others

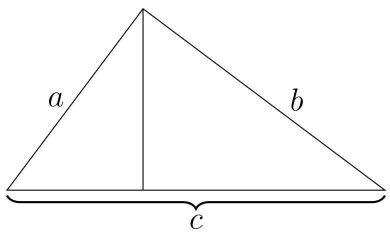
23. When rolling three fair, cubical dice, the probability of getting "doubles" (exactly two dice with the same value) is

    A: $\frac{5}{12}$ B: $\frac{5}{36}$ C: $\frac{1}{36}$ D: $\frac{5}{18}$ E: $\frac{1}{6}$

24. A pizza parlor offers a basic pizza for \$12.95. Additional toppings cost \$1.00 each. Customers are required to pay sales tax of 6% as well as a 15% service fee. [The basis for the service fee is the food charge only, not including the sales tax.] A group of friends wants to order a pizza, but they have only \$20 between them to pay for everything. How many toppings can they afford to put on their pizza?

A: 6 B: 5 C: 4 D: 3 E: none of the others

25. Consider a right triangle whose legs have lengths a and b, and whose hypotenuse has length c. Of all the points on the hypotenuse, the one closest to the opposite vertex lies what distance from that vertex?



A: $\dfrac{c}{ab}$ B: $\dfrac{2ab}{c}$ C: $\dfrac{a+b}{c}$ D: $\dfrac{ab}{c}$ E: none of the others

26. Mr. Scott, his sister, his son, and his daughter are tennis players. The people mentioned in the following facts refer to these four people.

1. The best player's twin and the worst player are of opposite sex.
2. The best player and the worst player are the same age.

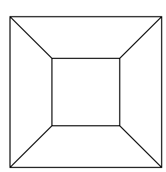
Which one of the four is the worst player?

A: Son B: Daughter C: Sister D: Mr. Scott
E. The given information is contradictory.

27. Two cards are drawn at random from a standard deck of 52. What is the probability that at least one of the drawn cards is an ace?

A: $\frac{33}{221}$ B: $\frac{2}{13}$ C: $\frac{25}{216}$ D: $\frac{103}{663}$ E: none of the others

28. A square of side length 1 is partitioned into four trapezoids and a central square of equal area as indicated in the diagram. The length of the lines connecting the corners of the outer square and the central square is



A: $\sqrt{2}\left(1-\frac{1}{\sqrt{5}}\right)$ B: $\frac{1}{\sqrt{5}}$ C: $1-\frac{1}{\sqrt{5}}$ D: $\frac{2}{\sqrt{3}}$ E: $\frac{\sqrt{5}-1}{\sqrt{10}}$

29. If $\tan\theta = \frac{1}{10}$, then the possible values of $\sin\theta$ are

A: $\frac{1}{\sqrt{101}}$ and $\frac{-1}{\sqrt{101}}$ B: $\frac{1}{\sqrt{101}}$ only C: $\frac{10}{\sqrt{101}}$ only
D: $\frac{10}{\sqrt{101}}$ and $\frac{-10}{\sqrt{101}}$ E: $\frac{1}{\sqrt{101}}$ and $\frac{10}{\sqrt{101}}$

30. A planar figure is said to be **tiled** by objects if it is exactly covered by them with no overlaps except at corners or along edges. You are given a set of one hundred 2×2 square tiles and one hundred 3×3 square tiles. Which of the following sized rectangles *cannot* be tiled using squares from the set? You are allowed to use both 2×2 squares and 3×3 squares in your tiling. You are also allowed to use only 2×2 squares or only 3×3 squares in your tiling.

    A: 5×6 B: 7×8 C: 8×8 D: 6×9 E: none of the others

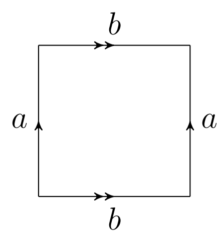
31. The letters a, b, c, d, e, f, g represent the numbers $1, 2, 3, 4, 5, 6, 7$ in a one to one fashion, though not necessarily in that order. Suppose that $a + b + c + d + e = 16$. Suppose also that there are integers x, y such that $gxy = 45$. Then $f =$

    A: 7 B: 6 C: 5 D: cannot be determined from the given information.
    E: none of the others

32. There are 30 students in a math class. There are two more females than males, and there are twice as many right-handed students as left-handed students. If there are 5 left-handed males, then how many left-handed females are in the class?

    A: 9 B: 7 C: 6 D: 5 E: none of the others

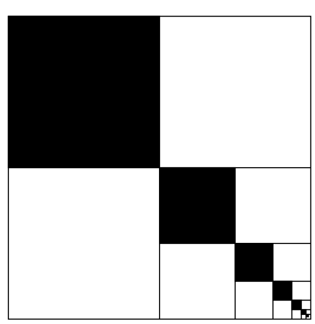
33. A square of flexible, stretchable material has its left and right sides glued together matching the arrows labeled a in the diagram and has its top and bottom sides glued together matching the arrows labeled b in the diagram. The result can be deformed into the shape of



    A: a torus (surface of a doughnut)
    B: a double torus (surface of a doughnut with two holes)
    C: a sphere D: a cylinder E: none of the others

34. The picture below shows a series of black squares inside a square of side length 1. The largest black square has side equal to half the side of the original square. Each succeeding black square has side equal to half the side of preceding one.

    What is the sum of the areas of the black squares?



    A: $\frac{1}{3}$ B: $\frac{\sqrt{2}}{3}$ C: $\frac{2}{5}$ D: ∞ E: none of the others

35. Two real numbers are such that the ratio of their difference, their sum, and their product is $1 : 9 : 100$. The product of the two numbers is

A: 200 B: 20 C: 1000 D: $100\sqrt{5}$ E: 500

36. Let S be the set of the following nine points in the plane:

$$\{\ (1,1),\ (1,2),\ (1,3),\ (2,1),\ (2,2),\ (2,3),\ (3,1),\ (3,2),\ (3,3)\ \}.$$

How many distinct lines pass through two or more points in S?

A: 20 B: 28 C: 36 D: 8 E: none of the others

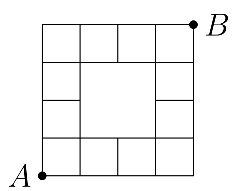
37. The product of two conjugate complex numbers is

A: always non-real B: always purely imaginary
C: always a negative real number D: always real E: none of the others

38. Suppose that a, b, c, d are non-zero real numbers and that the equation $ax^3 + bx^2 + cx + d = 0$ has three real roots. Then the sum of the reciprocals of these roots is

A: $-\dfrac{c}{d}$ B: $\dfrac{b}{a}$ C: $\dfrac{c}{d}$ D: $-\dfrac{b}{d}$ E: $-\dfrac{a}{b}$

39. The streets of the town of Blaise are laid out as indicated in the digram. How many ways are there to travel from the southwest corner A to the northeast corner B staying on the streets and traveling only north and east?



A: 16 B: 34 C: 42 D: 70 E: 10

40. Suppose that in the triangle $\triangle ABC$, $\cos A = \frac{\sqrt{2}}{2}, \cos B = \frac{4}{5}$, and the length of segment AB is 1. What this the length of segment BC?

A: $\frac{\sqrt{2}}{2}$ B: $\frac{5}{7}$ C: 1
D: There is not enough information to determine the length of BC.
E: none of the others

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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