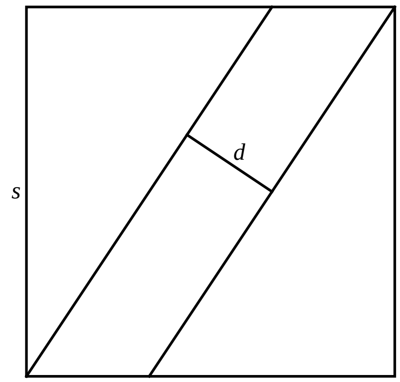
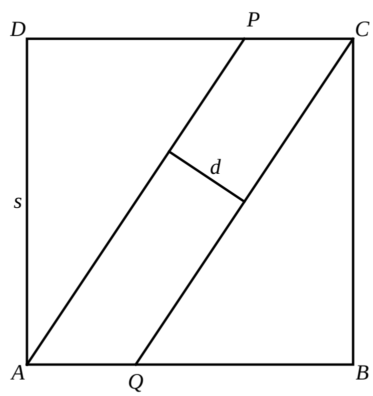
1. In the picture below, the two parallel cuts divide the square into three pieces of equal area. The distance between the two parallel cuts is d. The square has length s. Find and prove a formula that expresses s as a function of d.



Answer: $s = d\sqrt{13}$.



Proof. Let A, B, C, D, P, Q be as in the picture above. The triangle ADP has area $\frac{1}{3}s^2$, so $DP = \frac{2}{3}s$. By the Pythagorean Theorem,

$$AP = \sqrt{s^2 + \left(\frac{2}{3}s\right)^2} = \frac{s\sqrt{13}}{3}.$$

$AQCP$ is a parallelogram with base AP, height d, and area $\frac{1}{3}s^2$. Therefore

$$\frac{s^2}{3} = \frac{ds\sqrt{13}}{3},$$

and the result follows. □

2. Let S be a subset of $\{1, 2, 3, 4, \ldots, 10, 11\}$. We say that S is *lucky* if no two elements of S differ by 4 or 7.

   (a) Give an example of a lucky set with five elements.

   (b) Is it possible to find a lucky set with six elements? Explain why or why not.

   Answer:

   (a) $\{1, 3, 4, 6, 9\}$ is one possible answer.

   (b) No lucky set has six elements.

   Proof. Suppose, by way of contradiction, that S is a lucky set with six elements. First, consider the six sets

$$\{1,5\}, \quad \{2,9\}, \quad \{3,7\}, \quad \{4,11\}, \quad \{6,10\}, \quad \{8\}. \tag{1}$$

   A lucky set cannot contain more than one element of any of the sets in (**??**). Therefore, S must contain exactly one element of each set in (**??**). In particular, $8 \in S$.

   Next, consider the six sets

$$\{1\}, \{2,6\}, \quad \{3,10\}, \quad \{4,8\}, \quad \{5,9\}, \quad \{7,11\}. \tag{2}$$

   Again, S must contain exactly one element of each set in (**??**). In particular $1 \in S$. Therefore, 1 and 8 are both elements of S. However, this is a contradiction because $8 - 1 = 7$. □

3. Find polynomials $p(x)$ and $q(x)$ with real coefficients such that

   (a) $p(x) - q(x) = x^3 + x^2 - x - 1$ for all real x;

   (b) $p(x) > 0$ for all real x;

   (c) $q(x) > 0$ for all real x.

Solution: By inspection (with the aim of writing each function as sum of squares), let

$$p(x) = x^2\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}x^2 + 1 \text{ and } q(x) = x^4 + \left(x + \frac{1}{2}\right)^2 + \frac{7}{4}.$$

There are many ways to approach this problem. Another idea is to try to find polynomials f, g such that

$$f(x)^2 - g(x)^2 = x^3 + x^2 - x - 1.$$

This can be written as

$$(f(x) + g(x))(f(x) - g(x)) = (x^2 - 1)(x + 1) = (x + 1)^2(x - 1).$$

Solving the equations $f(x) + g(x) = (x + 1)^2$ and $f(x) - g(x) = x - 1$ gives

$$f(x) = \frac{1}{2}(x^2 + 3x), \quad g(x) = \frac{1}{2}(x^2 + x + 2).$$

This suggests taking $p(x) = f(x)^2$ and $q(x) = g(x)^2$, but this doesn't quite work since $p(-3) = 0$. However, we can finesse this by adding an arbitrary positive real number to both p and q. For example, we may take

$$p(x) = \frac{1}{4}(x^2 + 3x)^2 + \frac{1}{4} = \frac{1}{4}(x^4 + 6x^3 + 9x^2 + 1),$$
$$q(x) = \frac{1}{4}(x^2 + x + 2)^2 + \frac{1}{4} = \frac{1}{4}(x^4 + 2x^3 + 5x^2 + 4x + 5).$$

We note that without actually finding $p(x)$ and $q(x)$, we know such polynomials exist by observing that $p(x) = x^4 + x^3 + x^2 - x - 1 + N$ and $q(x) = x^4 + N$ will work if N is sufficiently large since both $x^4 + x^3 + x^2 - x - 1$ and x^4 are bounded below.

4. A permutation on $\{1, 2, 3, \ldots, n\}$ is a rearrangement of the symbols. For example 32154 is a permutation on $\{1, 2, 3, 4, 5\}$. Given a permutation $a_1a_2a_3 \ldots a_n$, an *inversion* is a pair of a_i and a_j such that $a_i > a_j$ but $i < j$; in other words, the pair is out of place. For example, 32154 has 4 inversions. Suppose you are only allowed to exchange adjacent symbols, show that the minimum number of exchanges required to put all the symbols in their natural positions (that is, $123 \ldots n$) of a given permutation is the number of inversions.

   Proof: We first show that the number of inversions in an upper bound by induction on n. The claim is clear for $n = 1, 2$ as there is only one permutation if $n = 1$ and two permutations if $n = 2$, thus the claim can be checked easily. Suppose the claim is true for $n = k$ for some fixed $k \geq 2$. Let $a_1a_2a_2 \ldots a_{n+1}$ be a permutation on $\{1, 2, 3, \ldots, n+1\}$. The inversions can be partitioned into two groups, those that involve the symbol $n+1$ and those that do not. The number of inversions in the first group is precisely the number of symbols to the right of $n+1$ in $a_1a_2a_2 \ldots a_{n+1}$. So after this many adjacent exchanges, $n+1$ is moved to the right-most position, its natural position. Now, the number of inversions in the new permutation is precisely the number of inversions in the second group of the old permutation. Since $n+1$ is in the correct position, we may apply the induction hypothesis on $\{1, 2, \ldots, n\}$.

   We now show that this number is the minimum. To see this, we note that each adjacent exchange either increases the number of inversions by exactly one or decreases the number by exactly one. Since the target permutation has 0 inversions, the number of "-1" required is at least the number of inversions.

5. Say that the number N is a nontrivial sum of consecutive positive integers if N can be expressed as a sum of 2 or more consecutive positive integers. Determine, with proof, the set of all integers N between 1000 and 2000 which are **not** nontrivial sums of consecutive integers.

Solution:

The sum of the d consecutive integers, $a, a+1, \ldots, a+(d-1)$ is $da+d(d-1)/2 = d(d+2a-1)/2$. Abbreviate this sum as $S(d,a)$.

Write $N = 2^k M$ where M is odd and k is a non-negative integer. Then $M \neq 2^{k+1}$ because 2^{k+1} is even. If $M > 2^{k+1}$, then $(M+1)/2 > 2^k$. Since $N = S(2^{k+1}, (M+1)/2 - 2^k)$, we see that N is a nontrivial sum of consecutive positive integers.

If $M < 2^{k+1}$, then $(M-1)/2 < 2^k$. Since $N = S(M, 2^k - (M-1)/2)$, we have that N is a nontrivial sum of consecutive positive integers in this case as well unless $M = 1$, that is, N is a power of 2.

Suppose $N = 2^k$ and $N = S(d,a)$. If p is an odd divisor of d, then $S(d,a) = p\left(\frac{d}{p}a + \frac{d(d-1)}{2p}\right)$, which is a multiple of p. It follows that d is a power of 2 and further that $N/d = a + (d-1)/2$ is not an integer. This is a contradiction because $N > d$.

This shows that no power of 2 is a nontrivial sum of consecutive positive integers.

Therefore the only integer between 1000 and 2000 which is not a nontrivial sum of consecutive positive integers is $2^{10} = 1024$.