Problem 1.

Let $x_1 = 0$, $x_2 = 1/2$ and for $n \ge 3$, let x_n be the average of x_{n-1} and x_{n-2}. Find a formula for $a_n = x_{n+1} - x_n, n = 1,2,3,\ldots$. Justify your answer.

Proof.

According to the definitions, we have

$$a_1 = x_2 - x_1 = \frac{1}{2}, \qquad x_{n+1} = \frac{x_n + x_{n-1}}{2}, \quad n = 2,3,4\ldots$$

Therefore we have the following results:

$$a_n = x_{n+1} - x_n = \frac{x_n + x_{n-1}}{2} - x_n = \frac{x_{n-1} - x_n}{2}; \quad a_{n-1} = x_n - x_{n-1}.$$

Based on the above results, we get

$$\frac{a_n}{a_{n-1}} = -\frac{1}{2}, \qquad a_n = \left(-\frac{1}{2}\right)a_{n-1} = \ldots = \left(-\frac{1}{2}\right)^{n-1} a_1$$

Problem 2.

Given a triangle ABC. Let h_a, h_b, h_c be the altitudes to its sides $a, b, c,$ respectively. Prove: $\frac{1}{h_a}+\frac{1}{h_b}>\frac{1}{h_c}$. Is it possible to construct a triangle with altitudes $7, 11$, and 20? Justify your answer.

Proof.

Let E be the area of triangle ABC. Then

$$2E = ah_a = bh_b = ch_c.$$

By the triangle Inequality,

$$a+b>c.$$

Therefore

$$\frac{2E}{h_a}+\frac{2E}{h_b}>\frac{2E}{h_c}.$$

The desired result follows upon dividing by $2E$.

No triangle with altitudes $7, 11$, and 20 is possible. This follows by using the result and noting that

$$\frac{1}{20}+\frac{1}{11}<\frac{1}{7}.$$

Problem 3.

Does there exist a polynomial $P(x)$ with integer coefficients such that $P(0)=1$, $P(2)=3$ and $P(4)=9$? Justify your answer.

Solution.

No. Since $P(0)=1$, $P(x)$ would be of the form $P(x)=1+xQ(x)$ where $Q(x)$ has integral coefficients. Since $3=P(2)=1+2Q(2)$, it follows that $Q(2)=1$. Thus $Q(x)$ is of the form $Q(x)=1+(x-2)R(x)$ where $R(x)$ has integral coefficients. Thus

$$P(x)=1+xQ(x)=1+x(1+(x-2)R(x))=1+x+x(x-2)R(x).$$

Thus, if $P(4)=9$, then $9=1+4+8R(4)$, so $R(4)=1/2$ which is impossible.

Problem 4.

Prove that if $\cos\alpha$ is rational and n is an integer, then $\cos n\alpha$ is rational. Let $\beta = \frac{\pi}{2010}$. Is $\cos\beta$ rational? Justify your answer.

Proof.

The proof is by induction on n. Assume that $\cos\alpha$ is rational. Let P_n be the statement that $\cos n\alpha$ is rational. The statement P_1 is true by the initial hypothesis. Note also that P_0 is true because $\cos 0 = 1$. Assume that P_m is true for $0 \le m \le n$. By the addition formula for $\cos x$,

$$\cos((n+1)\alpha) = 2\cos(n\alpha)\cos(\alpha) - \cos((n-1)\alpha)$$

By our induction assumption, all of the quantities on the right are rational. Therefore

$$\cos((n+1)\alpha)$$

is rational. The desired result follows by mathematical induction.

We prove that $\cos\beta$ is irrational. Suppose, by way of contradiction that $\cos\beta$ is rational. Then $\cos(335\beta)$ is rational by the previous part. However,

$$\cos(335\beta) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

is irrational.

Problem 5.

Let function $f(x)$ be defined as $f(x) = x^2 + bx + c$, where b, c are real numbers.

(A) Evaluate $f(1) - 2f(5) + f(9)$.

(B) Determine all pairs (b, c) such that $|f(x)| \le 8$ for all x in the interval [1, 9].

Proof

(A) Since

$$f(1) = 1 + b + c$$
$$f(5) = 25 + 5b + c$$
$$f(9) = 81 + 9b + c,$$

we can get

$$f(1) - 2f(5) + f(9) = 32$$

(B) Since $|f(x)| \le 8$ for all $x \in [1, 9]$, we have

$$|f(1)| \le 8; \qquad |f(5)| \le 8; \qquad |f(9)| \le 8.$$

Combining the above results and the results from part (A) we can get

$$\begin{aligned} 32 &= |f(1) - 2f(5) + f(9)| \\ &\le |f(1)| + 2|f(5)| + |f(9)| \le 32 \end{aligned}$$

This implies that

$$f(1) = f(9) = 8; \qquad f(5) = -8,$$

which means that

$$b + c + 1 = 8; \quad 9b + c + 81 = 8; \quad 5b + c + 25 = -8.$$

The only pair (b, c) that satisfies the condition when $b = -10$, and $c = 17$.