Problem 1.

Let $x_1 = 0$, $x_2 = 1/2$ and for $n \geq 3$, let $x_n$ be the average of $x_{n-1}$ and $x_{n-2}$. Find a formula for $a_n = x_{n+1} - x_n$, $n = 1, 2, 3, \ldots$. Justify your answer.

Proof.

According to the definitions, we have

$$a_1 = x_2 - x_1 = \frac{1}{2}, \quad x_{n+1} = \frac{x_n + x_{n-1}}{2}, \quad n = 2, 3, 4\ldots$$

Therefore we have the following results:

$$a_n = x_{n+1} - x_n = \frac{x_n + x_{n-1}}{2} - x_n = \frac{x_{n-1} - x_n}{2}; \quad a_{n-1} = x_n - x_{n-1}.$$ 

Based on the above results, we get

$$\frac{a_n}{a_{n-1}} = -\frac{1}{2}, \quad a_n = \left(-\frac{1}{2}\right)a_{n-1} = \ldots = \left(-\frac{1}{2}\right)^{n-1}a_1.$$
Problem 2.

Given a triangle $ABC$. Let $h_a, h_b, h_c$ be the altitudes to its sides $a, b, c$, respectively. Prove:

$$\frac{1}{h_a} + \frac{1}{h_b} > \frac{1}{h_c}.$$ Is it possible to construct a triangle with altitudes 7, 11, and 20? Justify your answer.

Proof.

Let $E$ be the area of triangle $ABC$. Then

$$2E = ah_a = bh_b = ch_c.$$

By the triangle Inequality,

$$a + b > c.$$

Therefore

$$\frac{2E}{h_a} + \frac{2E}{h_b} > \frac{2E}{h_c}.$$

The desired result follows upon dividing by $2E$.

No triangle with altitudes 7, 11, and 20 is possible. This follows by using the result and noting that

$$\frac{1}{20} + \frac{1}{11} < \frac{1}{7}.$$
Problem 3.

Does there exist a polynomial $P(x)$ with integer coefficients such that $P(0) = 1$, $P(2) = 3$ and $P(4) = 9$? Justify your answer.

Solution.

No. Since $P(0) = 1$, $P(x)$ would be of the form $P(x) = 1 + xQ(x)$ where $Q(x)$ has integral coefficients. Since $3 = P(2) = 1 + 2Q(2)$, it follows that $Q(2) = 1$. Thus $Q(x)$ is of the form $Q(x) = 1 + (x - 2)R(x)$ where $R(x)$ has integral coefficients. Thus

$$P(x) = 1 + xQ(x) = 1 + x(1 + (x - 2)R(x)) = 1 + x + x(x - 2)R(x).$$

Thus, if $P(4) = 9$, then $9 = 1 + 4 + 8R(4)$, so $R(4) = 1/2$ which is impossible.
Problem 4.

Prove that if \( \cos \alpha \) is rational and \( n \) is an integer, then \( \cos n \alpha \) is rational. Let \( \beta = \frac{\pi}{2010} \). Is \( \cos \beta \) rational? Justify your answer.

Proof.

The proof is by induction on \( n \). Assume that \( \cos \alpha \) is rational. Let \( P_n \) be the statement that \( \cos n \alpha \) is rational. The statement \( P_1 \) is true by the initial hypothesis. Note also that \( P_0 \) is true because \( \cos 0 = 1 \).

Assume that \( P_m \) is true for \( 0 \leq m \leq n \). By the addition formula for \( \cos x \),

\[
\cos((n+1)\alpha) = 2 \cos(n\alpha)\cos(\alpha) - \cos((n-1)\alpha)
\]

By our induction assumption, all of the quantities on the right are rational. Therefore \( \cos((n+1)\alpha) \) is rational. The desired result follows by mathematical induction.

We prove that \( \cos \beta \) is irrational. Suppose, by way of contradiction that \( \cos \beta \) is rational. Then \( \cos(335\beta) \) is rational by the previous part. However,

\[
\cos(335\beta) = \cos\left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}
\]

is irrational.
Problem 5.

Let function \( f(x) \) be defined as \( f(x) = x^2 + bx + c \), where \( b, c \) are real numbers.

(A) Evaluate \( f(1) - 2f(5) + f(9) \).

(B) Determine all pairs \((b, c)\) such that \(|f(x)| \leq 8\) for all \(x\) in the interval \([1, 9]\).

Proof

(A) Since
\[
\begin{align*}
f(1) &= 1 + b + c \\
f(5) &= 25 + 5b + c \\
f(9) &= 81 + 9b + c,
\end{align*}
\]
we can get
\[
f(1) - 2f(5) + f(9) = 32
\]

(B) Since \(|f(x)| \leq 8\) for all \(x \in [1, 9]\), we have
\[
|f(1)| \leq 8; \quad |f(5)| \leq 8; \quad |f(9)| \leq 8.
\]
Combining the above results and the results from part (A) we can get
\[
32 = |f(1) - 2f(5) + f(9)|
\]
\[
\leq |f(1)| + 2|f(5)| + |f(9)| \leq 32
\]
This implies that
\[
f(1) = f(9) = 8; \quad f(5) = -8,
\]
which means that
\[
b + c + 1 = 8; \quad 9b + c + 81 = 8; \quad 5b + c + 25 = -8.
\]
The only pair \((b, c)\) that satisfies the condition when \(b = -10\), and \(c = 17\).