1. Given a group of n people. An A-list celebrity is one that is known by everybody else (that is, $n-1$ of them) but does not know anybody. A B-list celebrity is one that is known by exactly $n-2$ people but knows at most one person.

   (a) What is the maximum number of A-list celebrities? You must prove that this number is attainable.

   (b) What is the maximum number of B-list celebrities? You must prove that this number is attainable.

   Sample Solution:

   (a) We claim that there can be no more than two A-list celebrities. Suppose both i and j are A-list celebrities. Since i is an A-list celebrity, i is known by j but i does not know j. But then j cannot be an A-list celebrity. Now, clearly having one A-list celebrity is attainable as given by the following situation: i is known by everybody else and i knows nobody.

   (b) The case $n = 1$ is not applicable. If $n = 2$, then it is easy to see that it is possible to have 0 or 1 or 2 B-list celebrities. Assume $n \geq 3$. We first show that it is possible to have three B-list celebrities. Suppose i knows j but not vice versa, j knows k but not vice versa, and k knows i but not vice versa. Moreover, everybody else knows i, j, k but i, j, k do not know them. Then i, j, k are B-list celebrities. We now claim that this is the maximum. Suppose i, j, k, x are all B-list celebrities. We consider two cases. The first case is two of them do not know each other, say i and j. Then everybody else, in particular, k and x know them. This is a contradiction as k and x are B-list celebrity. The second case is every pair from i, j, k, x contains at least a one-way acquaintance. Now since i is a B-list celebrity, we may assume j, k know i. (We assume nothing regarding i and x.) But x is a B-list celebrity. So at least one of j, k knows x. But they know i already, a contradiction.

2. A polynomial $p(x)$ has a remainder of $2, -13$ and 5 respectively when divided by $x+1, x-4$ and $x-2$. What is the remainder when $p(x)$ is divided by $(x+1)(x-4)(x-2)$?

Sample Solution:
Denote the remainder by $r(x)$ then $r(x)$ is quadratic. So we can write

$$\begin{aligned} & p(x) = (x+1)(x-4)(x-2)q(x) + r(x) = (x+1)(x-4)(x-2)q(x) + ax^2 + bx + c \\ @x = -1 \quad & 2 = a - b + c \\ @x = 4 \quad & -13 = 16a + 4b + c \\ @x = 2 \quad & 5 = 4a + 2b + c. \end{aligned}$$

One can readily solve this linear system to get $a = -2$, $b = 3$ and $c = 7$. So $r(x) = -2x^2 + 3x + 7$.

3. (a) Let x and y be positive integers satisfying $x^2 + y = 4p$ and $y^2 + x = 2p$, where p is an odd prime number. Prove: $x + y = p + 1$.

   (b) Find all values of x, y and p that satisfy the conditions of part (a). You will need to prove that you have found all such solutions.

Sample Solution:

(a) Subtract the two given equations to get

$$x^2 - y^2 - (x - y) = 2p,$$

which factors to

$$(x - y)(x + y - 1) = 2p.$$

Because p is a prime, we know that p divides exactly one of the numbers $x-y, x+y-1$. Similarly, 2 divides exactly one of these numbers.

Now $x^2 + y$ is even, so x and y have the same parity; i.e., $x - y$ is even. Now $x - y < x + y - 1$, so we cannot have $x - y = 2p$. Therefore, $x - y = 2$ and $x + y - 1 = p$.

(b) From part (a), $x - y = 2$ and $x + y = p + 1$. Solving for x and y, we find that

$$x = \frac{p + 3}{2}, \quad y = \frac{p - 1}{2}. \tag{1}$$

We plug these values back into the equation $y^2 + x = 2p$ to get

$$p^2 - 8p + 7 = 0.$$

This equation has roots $p = 7$ and $p = 1$. Now p is an odd prime, so $p = 7$. From equation (1), the values of x and y are determined by p. Therefore, there is only one set of values x, y, p that meet the conditions of part (a), and that is $x = 5, y = 3, p = 7$.

4. Let function $f(x, y, z)$ be defined as following:

$$f(x,y,z) = \cos^2(x-y) \;+\; \cos^2(y-z) \;+\; \cos^2(z-x), \quad x, y, z \in R.$$

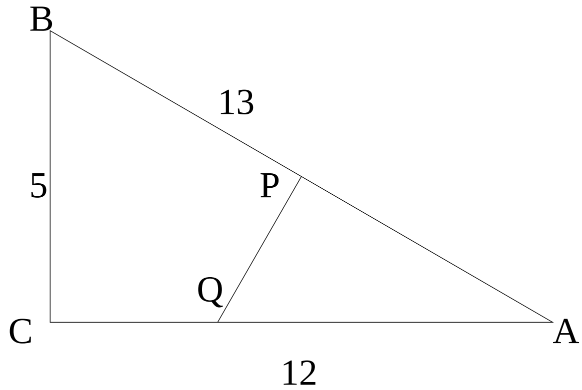
Find the minimum value and prove the result.

Sample Solution:

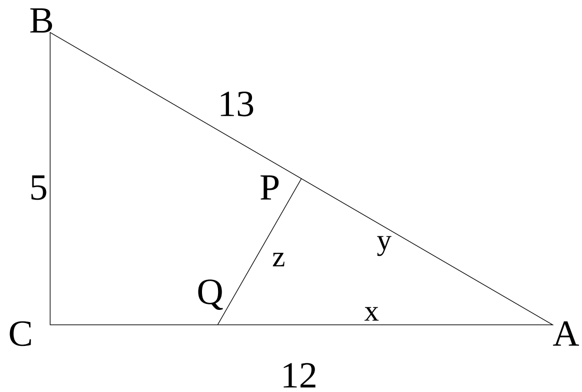
$$
\begin{aligned}
& f(x,\, y,\, z) \\
= \quad & \frac{1+\cos(2(x-y))}{2} + \frac{1+\cos(2(y-z))}{2} + \frac{1+\cos(2(z-x))}{2} \\
= \quad & \frac{3}{2} + \frac{1}{2}\,[\cos(2x)\,\cos(2y) + \cos(2y)\cos(2z) + \cos(2z)\cos(2x) \\
& + \sin(2x)\sin(2y) + \sin(2y)\sin(2z) + \sin(2z)\sin(2x)] \\
= \quad & \frac{3}{2} \;+\; \frac{1}{2}\left[\frac{1}{2}\,(\cos(2x)\;+\cos(2y)\;+\cos(2z))^2\right. \\
& + \frac{1}{2}\,(\sin(2x)+\sin(2y)+\sin(2z))^2 \\
& \left. -\frac{1}{2}\,\left(\cos^2(2x)+\cos^2(2y)+\cos^2(2z)+\sin^2(2x)+\sin^2(2y)+\sin^2(2z)\right)\right] \\
= \quad & \frac{3}{2} - \frac{3}{4} \;+\; \frac{1}{4}\,\left[(\cos(2x)+\cos(2y)+\cos(2z))^2 + (\sin(2x)+\sin(2y)+\sin(2z))^2\right] \\
\geq \quad & \frac{3}{4}
\end{aligned}
$$

The minimum value of $f(x,\, y,\, z)$ is $\frac{3}{4}$ which occurs at least at: $x = \frac{\pi}{3}$, $y = \frac{2\,\pi}{3}$, $z = \pi$.

5. In the diagram below, ABC is a triangle with side lengths $a = 5, b = 12, c = 13$. Let P and Q be points on AB and AC, respectively, chosen so that the segment $\overline{PQ}$ bisects the area of $\triangle ABC$. Find the minimum possible value for the length PQ.



Sample Solution: Let x, y, z be the lengths of AQ, AP, PQ respectively.



The area of triangle APQ is $\dfrac{xy \sin A}{2}$, and is one-half of the area of triangle ABC. Therefore $xy = 30/\sin A$. By the law of cosines,

$$z^2 = x^2 + y^2 - 2xy \cos A = (x-y)^2 + 2xy(1 - \cos A) = (x-y)^2 + \frac{60(1-\cos A)}{\sin A}.$$

This is minimized when $x = y$, and the minimum value satisfies

$$z^2 = \frac{60(1-\cos A)}{\sin A} = 60\frac{(1-\frac{12}{13})}{\frac{5}{13}} = 12$$

or $z = 2\sqrt{3}$.