FIFTY-FIRST ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 10, 2007

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.
2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.
3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.
4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.
5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(E) none of the others." If you believe none of the first four alternatives is correct, choose response (E).
6. Any scientific or graphing calculator is permitted on Part I. (Unacceptable machines include computers, PDAs, pocket organizers and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.) No other devices are permitted.
7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.
8. You may now open the test booklet and begin.

1. The value of $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{2006}{2007!}$ is

A: 0.9999 B: 1 C: $\frac{2006}{2007}$ D: $1 - \frac{1}{2007!}$* E: $\frac{1}{2007!}$

2. Let $f(x)$ be a function and suppose that $(7, 0)$ and $(12, 0)$ are the x-intercepts of $f(x)$. Let $g(x) = f(x^2 - 4x + 7)$. Find the x-intercepts of g.

A: $\{0, 4, 5, -1\}$* B: $\{1, 7, 6, 0\}$ C: $\{2, 6, 7, 1\}$ D: $\{3, 7, 8, 2\}$ E: none of the others

3. The sides of two cubes differ by two inches. Their volumes differ by 152 cubic inches. Find the length of the side of the larger cube.

A: 3 in B: 4 in C: 5 in D: 6 in* E: none of the others

4. Suppose that Sophie has four keys and that each key unlocks exactly one door. If she inserts the keys at random, one in each door, what is the probability that she will exactly unlock two doors?

A: 1/2 B: 1/3 C: 1/4* D: 1/5 E: none of the others

5. Let A be a two by two matrix. All entries of A are 1. Compute the entries of A^{100}.

A: 2^{98} B: 2^{99}* C: 2^{100} D: 2^{101} E: none of the others

6. The area of the region bounded by the x and y axes and a line whose slope is twice the value of its y-intercept $(0, b)$, where $b > 0$, is:

A: $\frac{b}{4}$* B: $\frac{b}{2}$ C: b D: $2b$ E: $4b$

7. 8 students play a game that lasts for 80 minutes but only allows 5 students to play at a time. If each of the 8 students plays the same length of time throughout the game, how many minutes does each student play?

A: 50* B: 40 C: 16 D: 10 E: none of the others

8. If $x^3 + x^2 + a\,x + b$ is divisible by $x^2 + x + 1$, then $b - a$ equals:

A: 2 B: 1 C: 0 D: -1* E: none of the others

9. If in a 9-sided regular polygon, a line is drawn from each of the 9 vertices to each of the other 6 vertices excluding the two neighboring ones, how many of these line segments will be drawn?

A: 18 B: 21 C: 27* D: 54 E: none of the others

10. AB and AC are two adjacent sides of a regular polygon. If angle BCA equals one-third of angle BAC, how many sides has the polygon?

A: 3 B: 4 C: 5* D: 10 E: none of the others

11. Let $p(x) = x^2 + bx + c$ and assume that b and c are non-zero integers. Which of the following are true?

I. If $p(x)$ has negative roots then $b > 0$ and $c > 0$.

II. If $c = 5$ and $p(x)$ has two integer roots then the number of possible values of b is 2.

III. If $p(x)$ has only one root then that root can be irrational.

A: I only B: II only C: III only D: I and II only* E: I and III only

12. Let p be the perimeter of triangle ABC. Let D be a point in the interior of the triangle. Let x, y and z be the lengths of DA, DB and DC, respectively. Which of the following are true?

I. $x + y + z > p/2$ II. $x + y + z < p$ III. $4(x + y + z) = p$

A: I only B: II only C: III only D: I and II only* E: I, II and III

13. Six students are going to sit in a row on a bench. Alex wants to sit next to Bob. Cassie does not want to sit with Debbie. Erin and Frank can sit anywhere. How many ways can these six students be seated?

A: 24 B: 96 C: 120 D: 144* E: 240

14. If $0 < A < \pi$ and $\sin A + \cos A = \frac{1}{5}$, then the value of $3\tan A$ is

A: -4* B: 4 C: -3 D: 3 E: 1

15. If $x + y = 1$ and $x^2 + y^2 = 2$ then the value of $x^5 + y^5$ is

A: 7 B: 6 C: 5.75 D: 5 E: 4.75*

16. In triangle ABC, the point D lies on BC and AD is the bisector of angle BAC. If $|AB| = c$, $|AC| = b$ and angle $CAD = x$, then $|AD|$ is

A: $\dfrac{bc\cos x}{b+c}$ B: $\dfrac{2bc\cos x}{b+c}$* C: $\dfrac{2bc\sin x}{b+c}$ D: $\dfrac{b\cos x + c\sin x}{b+c}$ E: $\dfrac{bc\cos x}{b+c}$

17. If the polynomial $p(x) = x^2 + bx + c$ has exactly one root r then $\dfrac{b}{c}$ is equal to

A: $-\dfrac{2}{r}$* B: $\dfrac{2}{r}$ C: $-\dfrac{2}{r^2}$ D: $\dfrac{2}{r^2}$ E: 1

18. Let T be an equilateral triangle of height h and S can be a square of side s. If T and S have the same area, then $\dfrac{h}{s}$ is

A: $3^{1/4}$* B: $2^{1/2}3^{1/4}$ C: $2\sqrt{3}$ D: $2\sqrt[4]{3}$ E: 1

19. A certain credit union uses four digit personal identification number (PIN) system. Each digit of a PIN can be any number from $\{0, 1, 2, \ldots, 9\}$ with the restriction that the first digit can not be 0. Which of the following best approximates the percentage of PIN's that contain at least one digit less than 5?

A: 7% B: 50% C: 93%* D: 94% E: 97%

20. Equilateral triangle ABC has area 1. Extend side AB to a point X so that B is the midpoint of AX. Extend BC so that C is the midpoint of BY, and extend CA so that A is the midpoint CZ. Which of the following are true?

I. Triangle XYZ is equilateral II. $|XY| = 7\,|AB|$ III. The area of triangle XYZ is 7.

A: I only B: I only C: II only D: I and III only* E: I, II and III

21. The lengths a, b and c of the three sides of a triangle satisfy $1 \le a \le 2 \le b \le 3 \le c \le 4$. The largest possible area of such a triangle is:

A: 3* B: 6 C: 9 D: 12 E: none of the others

22. If A_c the area of a square inscribed in a circle of radius r and A_s is the area of the square inscribed in a semicircle of the same circle, then $\dfrac{A_c}{A_s}$ is equal to:

A: $\dfrac{4}{3}$ B: 54 C: $\dfrac{5}{2}$* D: 5 E: none of the others

23. Let $f(x) = ax^2 + bx + c$ and suppose that $(5, 16)$ is the vertex and $f(1) = 32$. Find the value of c.

A: -41 B: 11 C: 32 D: 41* E: none of the others

24. Find the closed form for the sum $\displaystyle\sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}}$

A: 0 B: $\dfrac{1+2\sqrt{n+1}}{2}$ C: $\dfrac{\sqrt{n+1}-1}{\sqrt{n+1}}$* D: $\dfrac{1-\sqrt{n+1}}{\sqrt{n+1}}$ E: none of the others

25. The number of integer pairs that satisfy the equation $3x^2 - y^2 = 20 - 2xy$ is:

A: 8 B: 6 C: 4* D: 2 E: none of the others

26. Given that $0 \le \theta \le 2\pi$, the number of possible solutions to $8\cos^6(2\theta) - 7\cos^3(2\theta) - 1 = 0$ is:

A: 8 B: 7* C: 6 D: 4 E: 3

27. A regular deck of playing cards consists of four categories called suits. Each suit consists of 13 cards ranked using 3 face symbols (called King, Queen and Jack), a letter A (called Ace) and numbers 2, 3, . . . , 9. Suppose the value of each face card is 10. The value of Ace is 11 and the values of all remaining cards are the numbers they bear. What is the probability of drawing 3 cards without replacement whose sum is less than 12 and where each successive card drawn has a value less than the previous one.

A: $\dfrac{96}{132600}$ B: $\dfrac{192}{132600}$* C: $\dfrac{256}{132600}$ D: $\dfrac{384}{132600}$ E: $\dfrac{576}{132600}$

28. The smallest distance from the origin to the curve whose equation is given by

$$x^2 + y^2 = 24x - 10y - 153$$

is:

A: 13 B: 9* C: 4 D: 0 E: none of the others

29. Given that the quartic polynomial $p(x) = x^4 + Ax^3 + Bx^2 + Cx + D$ has repeated non-zero real roots at a and b. The value of $A^2 - 2B + \dfrac{C}{b} - \dfrac{2D}{a^2}$ is:

A: $2ab$ B: $-2ab$* C: $4ab$ D: $-4ab$ E: none of the others

30. A regular octagon, with largest possible area, was inscribed into a circle of radius 8. The area of the region outside the octagon but inside the circle is:

A: $64(\pi - 2\sqrt{2})$* B: $64(\pi - \sqrt{3})$ C: $64\left(\pi - \frac{\sqrt{2}}{2}\right)$ D: $64\left(\pi - \frac{\sqrt{3}}{2}\right)$
E: none of the others

31. The sequence a_n, $n \geq 0$, is defined as follows:
$a_0 = 1$, and $a_n = \begin{cases} a_{n-1}^2 + 1 & \text{if } a_{n-1} \text{ is odd} \\ 1 - a_{n-1}^2 & \text{if } a_{n-1} \text{ is even} \end{cases}$.
The value of $-2 - \dfrac{a_{2008}}{a_{2006}^2}$ is:

A: a_{2007}^2 B: a_{2006}^2* C: a_{2007} D: a_{2006} E: a_{2005}

32. If the equation

$$x^2 + 4x\sin\theta + \tan\theta = 0, \quad 0 < \theta < \frac{\pi}{2}$$

has repeated roots, then θ equals

A: $\dfrac{\pi}{12}$ B: $\dfrac{\pi}{6}$ C: $\dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$* D: $\dfrac{\pi}{6}$ or $\dfrac{\pi}{12}$ E: none of the others

33. The equation $x^2 - (2k+1)\,x + k^2 - 1 = 0$ has two roots: $\sin\theta$ and $\cos\theta$, where k and θ are both real constants. Then k equals:

A: -1* B: 0 C: $\frac{1}{2}$ D: 1 E: none of the others

34. The solution of $\sqrt{\log_3 x - 1} + \frac{1}{2}\frac{\log_3 x^3}{\log_3 \frac{1}{3}} + 2 > 0$ is:

A: $(3,\ 9]$ B: $[3,\ 9)$* C: $(9^{\frac{1}{3}},\ 9)$ D: $[3,\ \infty)$ E: none of the others

35. If the equation $\ln(x^2 + 5x) - \ln(x - a - 3) = 0$ has only one solution for x, then a:

A: equals 1 B: can be any value in $(-8,\ -3)$* C: can be any value in $(-\infty,\ -8)$ or $(-3,\ \infty)$ D: can be any value in $(-\infty,\ -8]$ or $[-3,\ \infty)$ E: can't be determined

36. The 1000^{th} term of the sequence:

$$\{1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, \cdots\} \text{ is}$$

A: 30 B: 31 C: 32* D: 33 E: none of the others

37. The curve $x^2 + 2\,y^2 = 3$ and the line $y = m\,x + b$ intersect for every real number m. The range for b is

A: $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ B: $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ C: $\left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right)$ D: $\left[-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right]$

E: none of the others*

38. Given $f(1) = 2$ where f is an odd function with period 3. Find $f\left((2007)^{2007} + 2005 + 2003 + \ldots + 1\right)$.

A: -2 B: 0 C: 1 D: 2* E: not enough information

39. If $(1 + x + x^2)^{100} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{200} x^{200}$ then $a_0 + a_2 + a_4 + a_6 + \cdots + a_{200}$ equals:

A: $\frac{3^{100} - 1}{2}$ B: 3^{100} C: $\frac{3^{100} + 1}{2}$* D: $\frac{3^{100}}{2}$ E: none of the others

40. Let $<n> = \frac{n(n+1)}{2}$. Find the closed form of $\sum_{k=1}^{n} <k>$.

A: $\frac{n}{3} <n-1>$ B: $\frac{n+1}{3} <n>$ C: $\frac{n+2}{3} <n>$* D: $\frac{n(n+1)}{2} <n>$

E: none of the others

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

DIRECTOR
Eddie Cheng
Oakland University

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