MMPC STUDENT CODE

FIFTY-FIRST ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

December 5, 2007

INSTRUCTIONS
(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. Let A be the point $(-1, 0)$, B be the point $(0, 1)$ and C be the point $(1, 0)$ on the xy-plane. Assume that $P(x, y)$ is a point on the xy-plane that satisfies the following condition

$$d_1 \cdot d_2 = (d_3)^2,$$

where d_1 is the distance from P to the line AB, d_2 is the distance from P to the line BC; and d_3 is the distance from P to the line AC.

Find the equation(s) that must be satisfied by the point $P(x, y)$.

Solution:

The equations for the three lines are:

$$\begin{aligned} &\text{AB:} \quad y = x + 1, \quad \text{or } x - y + 1 = 0, \\ &\text{BC:} \quad y = -x + 1, \quad \text{or } x + y - 1 = 0, \\ &\text{AC:} \quad y = 0. \end{aligned}$$

The three distances are

$$d_1 = \frac{|x - y + 1|}{\sqrt{2}}, \qquad d_2 = \frac{|x + y - 1|}{\sqrt{2}}, \qquad d_3 = |y|$$

Thus $d_1 \cdot d_2 = (d_3)^2$ becomes

$$y^2 = \frac{|(x - y + 1)\,(x + y - 1)|}{2},$$

which can be simplified to:

$$x^2 - (y - 1)^2 = 2y^2 \quad \text{or} \quad x^2 - (y - 1)^2 = -2y^2.$$

They are a hyperbola and a circle whose equations are given below.

$$x^2 - 3y^2 + 2y = 1, \quad \text{or} \quad x^2 + y^2 + 2y = 1.$$

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2. On Day 1, Peter sends an email to a female friend and a male friend with the following instructions:

   - If you're a male, send this email to 2 female friends tomorrow, including the instructions.
   - If you're a female, send this email to 1 male friend tomorrow, including the instructions.

   Assuming that everyone checks their email daily and follows the instructions, how many emails will be sent from Day 1 to Day 365 (inclusive)?

   Solution: Let's consider the results from the first 6 days.

At The End of Day	**# of Additional Email Sent to Male**	**# of Additional Email Sent to Female**
1=2(1)-1	1	1
2=2(1)	1	2
3=2(2)-1	2	2
4=2(2)	2	2^2
5=2(3)-1	2^2	2^2
6=2(3)	2^2	2^3

We thus see a pattern emerging. The number of additional emails sent to males and females at the end of day $2n-1$ is 2^{n-1} and the number of additional emails sent to males and females at the end of day $2n$ is 2^{n-1} and 2^n respectively.

Consequently the total number of emails sent to males at the end of the year is

$$\begin{aligned} & (1+1)+(2+2)+(2^2+2^2)+\cdots+(2^{181}+2^{181})+2^{182} \\ = & \ 2\left(1+2+2^2+\cdots+2^{181}\right)+2^{182} \\ = & \ 2(2^{182}-1)+2^{182} \\ = & \ 3(2^{182})-2. \end{aligned}$$

The total number of emails sent to females after 365 days is

$$\begin{aligned} & 1+(2+2)+(2^2+2^2)+\cdots+(2^{181}+2^{181})+(2^{182}+2^{182}) \\ = & \ 1+2\left(2+2^2+\cdots+2^{182}\right) \\ = & \ 1+4(2^{182}-1) \\ = & \ 4(2^{182})-3. \end{aligned}$$

So the total number of copies of the email sent is $7(2^{182})-5$.

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3. For every rational number $\frac{a}{b}$ in the interval $(0,1]$, consider the interval of length $\frac{1}{2b^2}$ with $\frac{a}{b}$ as the center, that is, the interval $\left(\frac{a}{b}-\frac{1}{4b^2}, \frac{a}{b}+\frac{1}{4b^2}\right)$. Show that $\frac{\sqrt{2}}{2}$ is not contained in any of these intervals.

Solution:
We show that $\frac{\sqrt{2}}{2} \geq \frac{1}{4b^2} + \frac{a}{b}$. Note that the number $|b^2 - 2a^2|$ is at least 1. If it were 0 it would contradict $\sqrt{2}$ being irrational.

Thus, $\frac{|b^2-2a^2|}{2b^2} \geq \frac{1}{2b^2}$. From this we get the inequality $|\frac{\sqrt{2}}{2} - \frac{a}{b}|(\frac{\sqrt{2}}{2} + \frac{a}{b}) \geq \frac{1}{2b^2}$

Hence, we get that

$|\frac{\sqrt{2}}{2} - \frac{a}{b}| \geq \frac{1}{2b^2}\frac{1}{\frac{\sqrt{2}}{2}+\frac{a}{b}} > \frac{1}{2b^2}\frac{1}{2} = \frac{1}{4b^2}$

From this last inequality we get that

$\frac{\sqrt{2}}{2} \geq \frac{1}{4b^2} + \frac{a}{b}$ showing that $\frac{\sqrt{2}}{2}$ lies outside the interval

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4. Let a and b be real numbers such that $0 < b < a < 1$ with the property that

$$\log_a x + \log_b x = 4\,\log_{ab} x - \left(\log_b(ab^{-1} - 1)\right)\left(\log_a(ab^{-1} - 1) + 2\,\log_a ab^{-1}\right)$$

for some positive real number $x \neq 1$. Find the value of $\dfrac{a}{b}$.

Solution:

$$\log_a x + \log_b x = 4\,\log_{a\,b} x - \left(\log_b(a\,b^{-1} - 1)\right)\left(\log_a(a\,b^{-1} - 1) + 2\,\log_a a\,b^{-1}\right)$$

can be rewritten as

$$\frac{1}{\log_x a} + \frac{1}{\log_x b} = \frac{4}{\log_x a\,b} - \left(\log_b(a\,b^{-1} - 1)\right)\left(\log_a(a\,b^{-1} - 1) + 2\,\log_a a\,b^{-1}\right).$$

Therefore,

$$\frac{\log_x a + \log_x b}{(\log_x a)(\log_x b)} = \frac{4}{\log_x a + \log_x b} - \left(\log_b(a\,b^{-1} - 1)\right)\left(\log_a(a\,b^{-1} - 1) + 2\,\log_a a\,b^{-1}\right).$$

Multiplying both sides by $(\log_x a)(\log_x b)(\log_x a + \log_x b)$, we get

$$\begin{aligned}(\log_x a + \log_x b)^2 &= 4\,(\log_x a)\,(\log_x b)\\ &\quad -(\log_x a + \log_x b)\left(\log_b(a\,b^{-1} - 1)\right)\left(\log_a(a\,b^{-1} - 1) + 2\,\log_a a\,b^{-1}\right).\end{aligned}$$

Hence,

$$\begin{aligned}(\log_x a + \log_x b)^2 &= 4\,(\log_x a)\,(\log_x b)\\ &\quad -(\log_x a)\,(\log_x b)\,(\log_x a + \log_x b)\left(\frac{\log_x(a\,b^{-1} - 1)}{\log_x b}\right)\left(\frac{\log_x(a\,b^{-1} - 1)}{\log_x a} + 2\frac{\log_x a\,b^{-1}}{\log_x a}\right).\end{aligned}$$

It follows that

$$(\log_x a - \log_x b)^2 + (\log_x a + \log_x b)\log_x(a\,b^{-1} - 1)(\log_x(a\,b^{-1} - 1) + 2\log_x(a\,b^{-1})) = 0$$

Hence,

$$\left(\log_x(a\,b^{-1})\right)^2 + (\log_x(ab)\log_x(a\,b^{-1} - 1)(\log_x(a\,b^{-1} - 1) + 2\log_x(a\,b^{-1})) = 0.$$

But this equation is not solvable for a/b.

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5. Find the largest positive constant λ such that

$$\lambda a^2 b^2 (a-b)^2 \leq (a^2 - ab + b^2)^3$$

is true for all real numbers a and b.

Solution: When $b = 0$, any positive constant λ works. Assume that $b \neq 0$ and divide b^6 to both sides of the inequality to get:

$$\lambda (\frac{a}{b})^2 \left(\frac{a}{b} - 1\right)^2 \leq \left((\frac{a}{b})^2 - \frac{a}{b} + 1\right)^3 .$$

Introduce $z = \frac{a}{b}$, then the problem becomes: finding the largest positive λ, such that:

$$(z^2 - z + 1)^3 - \lambda (z^2 - z)^2 \geq 0,$$

for any real number z. By further introduce $w = z^2 - z + 1$, the problem is changed to: finding the largest positive constant λ, such that

$$w^3 - \lambda (w-1)^2 \geq 0,$$

for any w, where $w = z^2 - z + 1 = (z - \frac{1}{2})^2 + \frac{3}{4} \geq \frac{3}{4}$.

By plotting both w^3 and $(w-1)^2$ for $w \geq \frac{3}{4}$, we can see that $w^3 > (w-1)^2$ is true for all $w \geq \frac{3}{4}$; and the smallest distance between them happens at $w = \frac{3}{4}$. So λ should satisfy:

$$(\frac{3}{4})^3 - \lambda (\frac{1}{4})^2 \geq 0,$$

which means that $\lambda \leq \frac{27}{4}$. The largest positive constant λ is $\frac{27}{4}$. (One way to check $27/4(w-1)^2 \leq w^3$ for $w >= 3/4$ is to observe $w^3 - 27(w-1)^2/4 = (w-3/4)(w-3)^2 >= 0$ for $w >= 3/4$.)

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(continued solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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