50th MMPC Part II Solution

(1) Suppose A, B and C are the angles of a triangle. Prove that

$$1 - 8\cos A\cos B\cos C = \sin^2(B-C) + (\cos(B-C) - 2\cos A)^2.$$

Solution:

$$\begin{aligned} RHS &= \sin^2(B-C) + \cos^2(B-C) - 4\cos A\cos(B-C) + 4\cos^2 A \\ &= 1 - 4\cos A\cos(B-C) + 4\cos^2 A \\ &= 1 - 4\cos A(\cos(B-C) - \cos A). \end{aligned}$$

Since $A + B + C = 180^\circ$ and $\cos(B+C) = -\cos(180^\circ - (B+C)) = -\cos(A)$, we have

$$\begin{aligned} RHS &= 1 - 4\cos A(\cos(B-C) + \cos(B+C)) \\ &= 1 - 4\cos A(2\cos B\cos C) \\ &= 1 - 8\cos A\cos B\cos C. \end{aligned}$$

(2) Let $x_1, x_2, \ldots, x_{100}$ be integers whose values are either 0 or 1.

(a) Show that

$$x_1 + x_2 + \cdots + x_{100} - (x_1x_2 + x_2x_3 + \cdots + x_{99}x_{100} + x_{100}x_1) \leq 50.$$

Solution: Let $S = x_1 + x_2 + \cdots + x_{100} - (x_1x_2 + x_2x_3 + \cdots + x_{99}x_{100} + x_{100}x_1)$. Out of the 100 integers n of them are equal 0 and the others are equal to 1. Then $S \leq 100 - n$. Since $x_1x_2 + x_2x_3 + \cdots + x_{99}x_{100} + x_{100}x_1 \geq 100 - 2n$, $S \leq 100 - n - (100 - 2\,n) = n$. Hence S is less than or equal to the average of $100 - n$ and n, which is 50.

(b) Give specific values for $x_1, x_2, \ldots, x_{100}$ that give equality.

Solution: $x_1 = x_3 = \ldots = x_{99} = 1$ and $x_2 = x_4 = \ldots = x_{100} = 0$ or $x_1 = x_3 = \ldots = x_{99} = 0$ and $x_2 = x_4 = \ldots = x_{100} = 1$.

(3) Let $ABCD$ be a trapezoid whose area is 32 square meters. Suppose the lengths of the parallel segments AB and DC are 2 meters and 6 meters, respectively, and P is the intersection of the diagonals AC and BD. If a line through P intersects AD and BC at E and F, respectively, determine, with a proof, the minimum possible area for quadrilateral $ABFE$.

Sketch of a solution: Through P, draw a line parallel to AB. Suppose the line intersects AD at M and BC at N. We first determine the area of $ABNM$ and then show it is at most the same as the area of $ABFE$.

Since $\Delta ABP \sim \Delta DPC$, $\frac{PB}{PD} = \frac{PA}{PC} = \frac{AB}{DC} = \frac{1}{3}$. It follows $\frac{BP}{DB} = \frac{AP}{AC} = \frac{1}{4}$. Since $\Delta AMP \sim \Delta ADC$, $\frac{MP}{DC} = \frac{AP}{AC} = \frac{1}{4}$. Thus, $MP = \frac{1}{4} \times 6 = 1.5$. Similarly, we can obtain $PN = 1.5$. Therefore, $MN = 3$.

Next, through P, draw a line perpendicular to AB. Suppose the line intersects AB and DC (or their extensions) at Q and R, respectively. Since the area of $ABDC$ is 32, it follows $QR = 8$. Since $\Delta PQB \sim \Delta PRD$, $\frac{PQ}{PR} = \frac{PB}{PD} = \frac{1}{3}$. It follows $PQ = 2$ and $PR = 6$. Now the area of $ABNM$ is $\frac{1}{2}\,(AB + MN) \times PQ = 5$.

To show the area of $ABNM$ is at most the same as the area of $ABFE$, without loss of generality, we suppose E is between D and M, and F is between B and N. We only need to show the area of ΔMPE is greater than the area of ΔFPN.

Extend DA and CB to meet at O. Since $MP = PN$, the area of ΔOMP equals the area of ΔOPN. Thus, the area of ΔOPF is less than the area of ΔOEP. Since ΔOPF and ΔOEP share a same altitude, it follows $EP > PF$.

Next, note that $\angle EPM \cong \angle FPN$. Since the area of $\Delta EMP = \frac{1}{2} \times MP \times EP \times \sin(\angle EPM)$ and the area of $\Delta FPN = \frac{1}{2} \times PN \times FP \times \sin(\angle FPN)$, the conclusion follows.

(4) Let n be a positive integer and x be a real number. Show that

$$\lfloor nx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to a. (For example, $\lfloor 4.5 \rfloor = 4$ and $\lfloor -4.5 \rfloor = -5$.)

Solution: Let x be expressed as $x = a + \epsilon$, where a is an integer and ϵ is so that $0 \leq \epsilon < 1$. Let n be a positive integer. We consider two cases. The first case is that $0 \leq \epsilon < \frac{1}{n}$. The second case is that $\frac{1}{n} \leq \epsilon < 1$.

Let's consider the first case. The condition $0 \leq \epsilon < 1$ implies that $n\epsilon < 1$. Hence, $\lfloor nx \rfloor = \lfloor na + n\epsilon \rfloor = na$.

Now $\epsilon + \frac{n-1}{n} < \frac{1}{n} + \frac{n-1}{n} = 1$. Therefore, $\lfloor x + \frac{n-1}{n} \rfloor = \lfloor a + \epsilon + \frac{n-1}{n} \rfloor = a$. Therefore, all the terms of the RHS are all equal to a. Thus, the RHS is na.

Let's consider now the second case. The condition $\frac{1}{n} \leq \epsilon < 1$ implies that $0 \leq n\epsilon - 1 < n - 1$. So we get the following for the LHS.

(a) $\lfloor nx \rfloor = \lfloor na + 1 + n\epsilon - 1 \rfloor = na + 1$ if $0 \leq n\epsilon - 1 < 1$.
(b) $LHS = na + 2$ if $1 \leq n\epsilon - 1 < 2$, etc...
(c) $LHS = na + n - 1$ if $n - 2 \leq n\epsilon - 1 < n - 1$.

Now, what about the RHS? The condition $0 \leq n\epsilon - 1 < 1$ implies that $\frac{1}{n} \leq \epsilon < \frac{2}{n}$. Consider the term of the RHS, $\lfloor a + \epsilon + \frac{n-2}{2} \rfloor$. The condition $\frac{1}{n} \leq \epsilon < \frac{2}{n}$ implies that $\frac{n-1}{n} \leq \epsilon + \frac{n-2}{2} < 1$. Hence, all the terms, except the last one,of the RHS must equal to a. The condition $\frac{1}{n} \leq \epsilon < \frac{2}{n}$ implies that $1 \leq \epsilon + \frac{n-1}{n} < \frac{n+1}{n}$. Hence the last term of the RHS must be $a + 1$. Therefore, summing gives us that $RHS = (n-1)a + a + 1 = na + 1$.

By similar reasoning, we get the following that

(a) $RHS = na + 2$ when $1 \leq n\epsilon - 1 < 2$, etc...
(b) $RHS = na + n - 1$ when $n - 2 \leq n\epsilon - 1 < n - 1$

Hence, we are done.

(5) A $3n$-digit positive integer (in base 10) containing no zero is said to be *quad-perfect* if the number is a perfect square and each of the three numbers obtained by viewing the first n digits, the middle n digits and the last n digits as three n-digit numbers is in itself a perfect square. (For example, when $n = 1$, the only *quad-perfect* numbers are 144 and 441.) Find all 9-digit *quad-perfect* numbers.

Solution: Denote the 9 digit number by N. Then N is a *quad-perfect* number iff we can represent it as $N = 10^6U^2 + 10^3V^2 + W^2$ where $U, V, W \in \{11, 12, \ldots, 19, 21, 22, \ldots, 29, 31\}$ and $N = (10^4a + 10^3b + 10^2c + 10d + e)^2$ where $a, b, c, d, \in \{0, 1, \ldots, 9\}$ and $e \in \{1, \ldots, 9\}$.

Observation 1:

$$(10^4a + 10^3b + 10^2c + 10d + e)^2 - 10^6U^2 = 10^3V^2 + W^2$$

$$\Rightarrow \Big[(10^3(10a + b - U) + 10^2c + 10d + e\Big]\Big[(10^3(10a + b + U) + 10^2c + 10d + e\Big] = 10^3V^2 + W^2.$$

R.H.S. of above equation is a 6-digit number. Second square bracket term is either a 5-digit number of a 6-digit number. Hence first square bracket term must be either a 1-digit number or a 2-digit number. So $c = 0$ and $U = 10a + b$.

Observation 2:

$$(10^4a + 10^3b + 10^2c + 10d + e)^2 - W^2 = 10^6U^2 + 10^3V^2$$

$$\Rightarrow \Big[10^3U + 10d + e - W\Big]\Big[10^3U + 10d + e + W\Big] = 10^3(10^3U^2 + V^2).$$

R.H.S. of above equation implies L.H.S. is divisible by 1000. Hence $(10d+e+W)(10d+e-W)$ is divisible by 1000.

Note that $12 \leq 10d+e+W \leq 130$ and $10d+e-W \leq 88$. Clearly $(10d+e+W)(10d+e-W)$ is divisible by 1000 if $W = 10d + e$. Let's consider this case first.

Hence $[10^3U][10^3U + 2W] = 10^3(10^3U^2 + V^2)$. So $2UW = V^2 \Rightarrow V = \sqrt{2UW}$, hence at least one of U or W must be even.

The only possible choice of odd U is $U = 11$ and $U = 13$ since if $U = 15$ then $V = \sqrt{30W}$, so $W = 30$ but then $W^2 = 900$ and so N will contain zeros hence it is not a *quad-perfect* number. For $U \geq 17$, U odd, choice of W will need to be greater than 31, not possible.

If $U = 11$ then $V = \sqrt{22W}$ hence $V = W = 22$ so the number is $\boxed{121484484}$. By symmetry, the number $\boxed{484484121}$ also works.

If $U = 13$ then $V = W = 26$, so the resulting *quad-perfect* numbers are $\boxed{169676676}$ and $\boxed{676676169.}$

Now let's consider U even:
If $U = 12$ then $V = 2\sqrt{6W}$ hence $W = 24 \Rightarrow V = 24$. So the resulting *quad-perfect* numbers are $\boxed{144576576}$ and $\boxed{576576144.}$
If $U = 14$ then $V = 2\sqrt{7W}$ hence $W = 28 \Rightarrow V = 28$. So the resulting *quad-perfect* numbers are $\boxed{196784784}$ and $\boxed{784784196.}$
If $U = 16$ then $V = 4\sqrt{2W}$ hence $W = 18 \Rightarrow V = 24$. So the resulting *quad-perfect* numbers are $\boxed{256576324}$ and $\boxed{324576256.}$
We don't need to consider $U = 18, 22, 24, 26$ and 28 for they have already been included above by symmetry.

Suppose $W = 10d + e$ is not true. Since $(10d + e + W)(10d + e - W)$ is divisible by 1000, $(10d + e + W)$ and $(10d + e - W)$ must contain, collectively, 3 2's and 3 5's as factors. We first claim each must a factor of 5.

- $(10d + e + W)$ is divisible by 5. Suppose not. Then $(10d + e - W)$ is divisible by 125 which is impossible as $1 \leq 10d + e - W \leq 88$.
- $(10d + e - W)$ is divisible by 5. Suppose not. Then $(10d + e + W)$ is divisible by 125. Hence $(10d + e + W) = 125$, an odd number. So $10d + e - W$ is divisible by 8, an even number. But their difference is $2W$, a contradiction.

We now claim each must contain a factor of 2. Suppose not. Then exactly one of them is not divisible by 2. So one of them is even and the other is odd. But their difference is $2W$, a contradiction.

Hence d, e and W satisfy the following 3 conditions:

(a) $(e - W)(\text{mod } 10)$=0
(b) $(e + W)(\text{mod } 10)$=0
(c) $\left(d + \frac{e-W}{10}\right)\left(d + \frac{e+W}{10}\right)$ (mod 10)=0

Since $e \in [1, 9]$. For (1) and (2) to hold, we must have $e = 5$. Hence, $W = 15$ or $W = 25$.

Case 1: $W = 15$**,** $e = 5$
Condition (3) becomes $(d - 1)(d + 2)$ (mod 10) =0. So the only possible choice is $d = 3, 6$ and 8.
When $d = 3$:

$$\left[10^3U + 10d - 10\right]\left[10^3U + 10d + 20\right] = 10^3(10^3U^2 + V^2)$$
$$\Rightarrow \quad 70U + 1 = V^2.$$

Since $U \geq 11$, $70U + 1 \geq 771$, thus the only possible choice for V is 29. One can verify that with $V = 29 \Rightarrow U = 12$. So $\boxed{144841225}$ is a *quad-perfect* number.
Similarly, when $d = 6, 8$ the resulting equations are $130U + 4 = V^2$ and $170U + 7 = V^2$ for which clearly no choice of allowable V is possible.

Case 2: $W = 25$**,** $e = 5$
Condition (3) becomes $(d - 2)(d + 3)$ (mod 10) =0. So the only possible choice is $d = 2$ and 7.
When $d = 2, 7$, the resulting equations are $50U = V^2$ and $150U + 5 = V^2$ for which clearly no choice of allowable V is possible.

So in total, there are 11 *quad-perfect* 9-digit numbers.