FIFTIETH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 11, 2006

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.

3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.

5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "E: none of the above." If you believe none of the first four alternatives is correct, choose response E.

6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.

7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.

8. You may now open the test booklet and begin.

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1. The number of pairs of integers (a, b) with $-500 \le a \le 500$ and $-500 \le b \le 500$ that satisfy the equation $3a + b = \dfrac{10ab}{2a+b}$ is

   A: 759 B: 832* C: 833 D: 999 E: 1001

2. Maddox has placed all the songs in his iTunes library into one of 10 playlists. Suppose 1/3 of his entire library is repeated exactly once and the number of non-repeating songs in the 10 playlists are $18, 10, 10, 5, 5, 15, 7, 4, 8, 20$, respectively. How many songs are in Maddox's iTunes library?

   A: 102 B: 136 C: 153* D: 175 E: none of the above

3. Greg, James, Lisa, Robert, Allison and Eric have each picked a positive integer.

Greg says:	Our numbers are neither all odd nor all even.
James says:	My number is twice that of Greg's.
Lisa says:	My number is the sum of the others.
Robert says:	My number is the sum of James' and Greg's.
Allison says:	My number is an odd multiple of Robert's.
Eric says:	My number is the product of the others, excluding Lisa's.

   Of the 6 numbers, how many are odd? Assume they tell the truth.

   A: 2 B: 3 C: 4* D: 5 E: cannot be determined from the given information

4. Given $f(x) = ax + b$, $a \ne 0$. Find $g(2006!)$, where, for each positive integer n, $g(n)$ is defined by

   $$g(n) = f^{-1}((-1)^{n+1}af^{-1}((-1)^n af^{-1}(\cdots f^{-1}((-1)^3 af^{-1}((-1)^2 af^{-1}(0))))))$$

   A: $-\dfrac{b}{a}$* B: 0 C: $\dfrac{b}{a}$ D: $\dfrac{2006b}{a}$ E: $\dfrac{2006!b}{a}$

5. The number of pairs of integers (x, y) that satisfy the following two equations:
   $\begin{cases} \cos(xy) = x \\ \tan(xy) = y \end{cases}$ is

   A: 1* B: 2 C: 4 D: 6 E: 8

6. A triangle, on the xy-plane, is to be formed using the x-axis, a line with positive slope m passing through the origin, and a line with slope $2m$ and y-intercept b. If the area of the triangle is $3b$, then the relationship that m and b must satisfy is

   A: $m = 3b$ B: $b = 3m$ C: $m = 6b$ D: $b = m$ E: $b = 12m$*

7. How many distinct four-digit numbers can be formed by rearranging the four digits in 2006?

   A: 4 B: 6* C: 10 D: 12 E: 24

8. Let a, b and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

   A: 0 B: 1* C: 16 D: 49 E: 65

9. Let f be a linear function for which $f(7) - f(3) = 16$. What is $f(13) - f(3)$?

   A: 16 B: 20 C: 40* D: 52
   E: cannot be determined from the given information

10. Triangle ABC is an equilateral triangle with side length 2. Suppose M is the midpoint of $\overline{AC}$ and D is a point on the extension of $\overline{BC}$ such that C is the midpoint of $\overline{BD}$. Then the area of triangle CDM is

    A: $\frac{\sqrt{3}}{4}$ B: $\frac{1}{2}$ C: $\frac{\sqrt{3}}{2}$* D: 2 E: $\sqrt{3}$

11. Alice, Bob, Christine and David are going to drive together to a movie theater. The car they are using has four standard seats: one driver's seat, one front passenger seat, one left rear seat, and one right rear seat. If Alice and David are the only ones who can drive, then the number of possible seating arrangements is

    A: 2 B: 4 C: 6 D: 12* E: 24

12. The graph of the polynomial function $P(x) = x^5 + a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e$ has five distinct x-intercepts, one of which is $(0, 0)$. Which of the coefficients cannot be zero?

    A: a B: b C: c D: d* E: e

13. Suppose a, b, $a - b$ and $a + b$ are all prime numbers. Which of the following are true?

    I. $a - b$ must be even II. $b = 2$ III. $a = 2$

    A: I only B: II only* C: III only D: I and II only E: I and III only

14. Each of the distinct numbers a and b exceeds its reciprocal by 1. What is $a + b$?

    A: 1* B: 2 C: $\sqrt{5} - 1$ D: $\sqrt{5}$ E: $1 + \sqrt{5}$

15. An equilateral triangle has one of his sides coinciding with the diameter of a unit circle. The area of the region inside the triangle but outside the circle is

A: $\frac{1}{2} - \frac{\pi}{12}$ B: $\frac{\pi}{3} - \frac{1}{2}$ C: $\sqrt{3} - \frac{\pi}{3}$ D: $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$ E: $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$*

16. If $x = 2$ and $x = -3$ are the only real roots of some 4th degree polynomial, which of the following CANNOT possibly be that polynomial?

A: $x^4 + (k+2)x^3 + (k^2+k-5)x^2 + (k^2-6k-6)x - 6k^2$
B: $x^4 + (2k+3)x^3 + (4k-4)x^2 - (10k+12)x - 12k$*
C: $2x^4 + (k+3)x^3 + (2k-10)x^2 - 5(k+1)x - 6(k+1)$
D: $2x^4 + (k+3)x^3 + (k-9)x^2 - (6k+4)x - 12$
E: none of the above

17. What is the largest possible range of values of k such that $2\cos^4 x - \sin^4 x + k = 0$ has at least one solution?

A: $[-1, 1]$ B: $[-1, 2]$ C: $[-2, -1]$ D: $[-2, 1]$* E: $[-2, 2]$

18. Let $n \geq 4$. How many diagonals does a regular $5n$-sided polygon have?

A: $\frac{n^2(5n-7)}{2}$ B: $\frac{5n(5n-3)}{2}$* C: $n(n^2-7)$ D: $25n^2 - 5$ E: $25n^2 - 10$

19. Suppose $f(x) = x^2 - 4x - 1$ and $g(x) = |x|$. Let $h(x) = g(f(x)) + 10$. Find the range of $h(x)$.

A: $\{y : y > -5\}$ B: $\{y : y > 0\}$ C: $\{y : y > 5\}$ D: $\{y : y > 10\}$
E: $\{y : y \geq 10\}$*

20. Let $P = xy$ and $x + 3y = a$. Then the largest possible value for P is

A: $\frac{a^2}{12}$* B: $\frac{a^2-2}{12}$ C: $\frac{a^2}{10}$ D: $\frac{a^2+2a}{12}$ E: $\frac{-a^2}{4}$

21. Let n be a positive integer. How many nonnegative integer solutions does the equation $x + y + z = n$ have?

A: $\frac{n(n+1)(n+2)}{6}$ B: $\frac{(n+2)(n+1)}{2}$* C: $\frac{3n(n+1)}{2}$ D: $n+2$ E: $3n$

22. Consider the set $\{1, 2, 3, 4\}$ and suppose that we randomly select one of its subsets. What is the probability that the elements of the subset sum to a value strictly less than 5? Assume, by default, the sum of elements in the empty set is zero.

A: $\frac{6}{15}$ B: $\frac{6}{16}$ C: $\frac{7}{16}$* D: $\frac{9}{16}$ E: $\frac{10}{16}$

23. Call a positive integer *one-derful* if its digits are all 1s. How many positive integers less than 10 are factors of some one-derful number?

A: 1 B: 2 C: 3 D: 4* E: none of the above

24. For how many integers k, between 1 and 1000, does $x^2 + 3x - k$ factor into the product of linear factors with integer coefficients?

A: 30* B: 31 C: 32 D: 1000 E: none of the above

25. Find the digit k such that the base five numeral $3k23$ represents the same number as the base eight numeral 666.

A: 0 B: 1 C: 2* D: 3 E: none of the above

26. Suppose $A(-1, -1), B(1, 3)$ and $C(5, 1)$ are points in the plane. If D is the midpoint of $\overline{BC}$ then the slope of the line containing $\overline{AD}$ is

A: $\frac{1}{2}$ B: $\frac{3}{4}$* C: $\frac{4}{3}$ D: 2 E: none of the above

27. Suppose $f(x)$ is a polynomial of degree 5 and the sum of its five roots is 20. Determine the sum of the roots of $f(x - 2)$.

A: 10 B: 18 C: 20 D: 22 E: 30*

28. An icosahedron is a regular polyhedron with 20 triangular faces. How many edges does it have?

A: 12 B: 20 C: 24 D: 30* E: 60

29. The distance from $(0, 13)$ to the line $-2x + 3y = 0$ is

A: $\sqrt{13}$ B: $2\sqrt{13}$ C: $3\sqrt{13}$* D: $4\sqrt{13}$ E: none of the above

30. If Patrick, who is 6 feet tall, walked around the Earth at the equator, how many feet farther would his head travel than his feet?

A: $\frac{6}{\pi}$ ft B: $\frac{3}{\pi}$ ft C: 6π ft D: 12π ft* E: more than 1000 ft

31. If L is the tangent line to the circle $(x+2)^2+y^2=4$ at $P(-3,\sqrt{3})$, then the y-intercept of L is

A: $(0, \frac{3\sqrt{3}}{2})$ B: $(0, \frac{4\sqrt{3}}{3})$ C: $(0, \frac{5\sqrt{3}}{4})$ D: $(0, 2\sqrt{3})$*
E: none of the above

32. Which polar equation is equivalent to the Cartesian equation $(x^2+y^2)^2=x^2-y^2$?

A: $r^2=\cos(2\theta)$* B: $r^2=\sin(2\theta)$ C: $r^2=\cos\theta$ D: $r^2=\sin\theta$
E: none of the above

33. If $0<\theta<\pi$ and $2\sin^2\theta+5\sin\theta-3=0$, then θ, in radians, must be

A: $\frac{\pi}{12}$ B: $\frac{\pi}{6}$ C: $\frac{\pi}{3}$ D: cannot be determined from the given information*
E: none of the above

34. Let x and y be numbers in the open interval (0, 1). Suppose there exists a positive number a different from 1 such that $\log_x a+\log_y a=4\log_{xy} a$. Which of the following statements are necessarily true?

I. $(\log_a x+\log_a y)^2=4\log_a x\log_a y$ II. $x=y$ III. $\log_{x^2} a=\log_x \sqrt{a}$

A: I only B: II only C: I and II only D: II and III only E: I, II and III*

35. The operation $\star$ is defined by: $x\star y=\dfrac{x^2+y^2}{x+y}$. If $(x\star x)\star 3=5$ then which of the following could be a value for x?

A: -3 B: -2 C: 1 D: 6* E: none of the above

36. Consider the equations $x^2+a\,x+b=0$ and $x^2+c\,x+a=0$, where none of a, b, c is zero. If the roots of the first equation are twice those of the second, respectively, then $\frac{b}{c}$ is equal to

A: $\frac{1}{8}$ B: $\frac{1}{4}$ C: 2 D: 4 E: 8*

37. Suppose a function f has the property that $f\left(\frac{ay}{x}\right) = \frac{1}{a} f\left(\frac{x}{y}\right)$, for all $a, x, y > 0$. Then the value of $f\left(\frac{4}{3}\right)$ is

A: 0* B: $\frac{3}{4}$ C: 1 D: $\frac{4}{3}$ E: cannot be determined from the given information

38. The sequence a_n, $n \geq 0$, is defined as follows: $a_0 = 1$, and $a_n = f(a_{n-1})$ for all $n \geq 1$, where $f(k) = \begin{cases} k-2 & \text{if } k \geq 0 \\ k+5 & \text{if } k < 0 \end{cases}$. The value of a_n for $n = 2006^{2006}$ is

A: 0 B: 1 C: 3 D: 4* E: none of the above

39. Which of the following is equal to

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{2006^2}\right)?$$

A: $\frac{1}{2006}$ B: $\frac{3}{2006}$ C: $\frac{2005}{4012}$ D: $\frac{2007}{4012}$* E: none of the above

40. An arch is built in the form of an arc of a circle and is subtended by a chord 30 ft long. If a cord 17 ft long subtends half that arc, what is the radius of the circle?

A: 8 ft B: 16 ft C: 18 ft D: 18 ft $\frac{3}{4}$ in* E: none of the above

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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