MMPC STUDENT CODE

FIFTIETH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

December 6, 2006

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. Suppose A, B and C are the angles of a triangle. Prove that

$$1 - 8\cos A\cos B\cos C = \sin^2(B-C) + (\cos(B-C) - 2\cos A)^2.$$

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☐ Check here if this solution is continued on additional paper that you are inserting.

2. Let $x_1, x_2, \ldots, x_{100}$ be integers whose values are either 0 or 1.

   (a) Show that

$$x_1 + x_2 + \cdots + x_{100} - (x_1x_2 + x_2x_3 + \cdots + x_{99}x_{100} + x_{100}x_1) \leq 50.$$

   (b) Give specific values for $x_1, x_2, \ldots, x_{100}$ that give equality.

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3. Let $ABCD$ be a trapezoid whose area is 32 square meters. Suppose the lengths of the parallel segments AB and DC are 2 meters and 6 meters, respectively, and P is the intersection of the diagonals AC and BD. If a line through P intersects AD and BC at E and F, respectively, determine, with a proof, the minimum possible area for quadrilateral $ABFE$.

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4. Let n be a positive integer and x be a real number. Show that

$$\lfloor nx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to a. (For example, $\lfloor 4.5 \rfloor = 4$ and $\lfloor -4.5 \rfloor = -5$.)

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5. A $3n$-digit positive integer (in base 10) containing no zero is said to be *quad-perfect* if the number is a perfect square and each of the three numbers obtained by viewing the first n digits, the middle n digits and the last n digits as three n-digit numbers is in itself a perfect square. (For example, when $n = 1$, the only *quad-perfect* numbers are 144 and 441.) Find all 9-digit *quad-perfect* numbers.

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(continued solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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