49th MMPC Part II Exam with solutions

1. Two perpendicular chords intersect in a circle. The lengths of the segments of one chord are 3 and 4. The lengths of the segments of the other chord are 6 and 2. Find the diameter of the circle.

   Solution. Let C_1 be the chord whose segment lengths are 3 and 4. Let C_2 be the chord whose segment lengths are 2 and 6. The perpendicular bisectors of the chords C_1 and C_2 intersect at the center of the circle. The perpendicular bisectors further divide the chords; C_1 now has segments of length 3, .5, and 3.5; C_2 now has segments of length 2, 2, and 4. To finish the problem consider the right triangle given by connecting the center of the circle to one of the ends of C_1 or C_2. Then use the Pythagorean Theorem to find the radius of the circle. One way to do this is

   $r^2 = 4^2 + (.5)^2 = 65/4$,
   $r = \sqrt{65}/2$,
   $d = \sqrt{65}$.

2. Determine the greatest integer that will divide 13,511, 13,903 and 14,589 and leave the same remainder.

   Solution. If the integers $a = 13,511$, $b = 13,903$ and $c = 14,589$ have the same remainder r after division by an integer d, then

   $$a = \alpha d + r, \quad b = \beta d + r, \quad \text{and} \quad c = \gamma d + r,$$

   where α, β, and γ are quotients. The differences

   $$a - b = (\alpha - \beta)d, \quad a - c = (\alpha - \gamma)d, \quad \text{and} \quad b - c = (\beta - \gamma)d$$

   are all divisible by d. Moreover, since

   $$(a - b) - (a - c) + (b - c) = 0,$$

   any common divisor d of two of the the differences is a divisor of the third. Hence the GCD of any pair of differences is the greatest integer leaving the same remainder when divided into all three of the original numbers a, b, and c.

   Thus we seek the the GCD or the two differences

   $$b - a = 13,903 - 13,511 = 392 = 7^2 2^3$$

   and

   $$c - b = 14,589 - 3,903 = 686 = 7^3 2.$$

   Hence the desired answer is $7^2 2 = 98$.

3. Suppose A, B and C are the angles of the triangle. Show that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

Solution: Since $A = \pi - (B + C)$, $\cos A = -\cos(B + C) = \sin B \sin C - \cos B \cos C$. Therefore,

$$\begin{aligned} LHS &= (\cos A + \cos B \cos C)^2 + \cos^2 B + \cos^2 C - (\cos B \cos C)^2 \\ &= (\sin B \sin C)^2 + \cos^2 B\,(1 - \cos^2 C) + \cos^2 C \\ &= (\sin B \sin C)^2 + \cos^2 B\,(1 - \cos^2 C) + \cos^2 C - 1 + 1 \\ &= (\sin B \sin C)^2 - (1 - \cos^2 C)(1 - \cos^2 B) + 1 \\ &= (\sin B \sin C)^2 - (\sin B \sin C)^2 + 1 \\ &= 1. \end{aligned}$$

4. Given the linear fractional transformation $f_1(x) = \dfrac{2x - 1}{x + 1}$. Define

$$f_{n+1}(x) = f_1(f_n(x)) \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

It can be shown that $f_{35} = f_5$.

(a) Find a function g such that $f_1(g(x)) = g(f_1(x)) = x$.

(b) Find f_{28}.

Solution.(Sketch.) (a) $g(x) = (x + 1)/(2 - x)$. (b) By part (a) we see that g is the inverse of f. It can be shown that $f_{30}(x) = x$ by applying g five times to both sides of the functional equation $f_{35} = f_5$. Thus $f_{28} = g_2$ where we define $g_2(x) = g(g(x))$. A calculation shows that $g_2(x) = 1/(1 - x)$.

5. Suppose a is a complex number such that $a^{10} + a^5 + 1 = 0$.

Determine the value of $a^{2005} + \frac{1}{a^{2005}}$.

Solution. Multiplying $a^{10} + a^5 + 1 = 0$ by a^5 gives $a^{15} + a^{10} + a^5 = 0$. Subtracting $a^{10} + a^5 + 1 = 0$ from $a^{15} + a^{10} + a^5 = 0$, we obtain $a^{15} = 1$. Since $a \neq 0$, dividing $a^{10} + a^5 + 1 = 0$ by a^5 yields $a^5 + \frac{1}{a^5} = -1$. Squaring the last equation leads to $a^{10} + \frac{1}{a^{10}} = -1$. Now, $a^{2005} + \frac{1}{a^{2005}} = (a^{15})^{133} a^{10} + \frac{1}{(a^{15})^{133} a^{10}} = a^{10} + \frac{1}{a^{10}} = -1$.