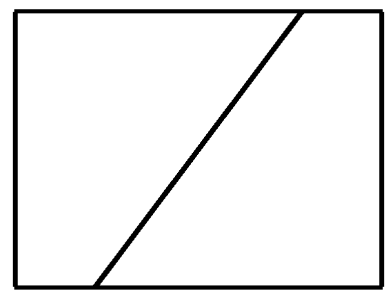
FORTY-EIGHTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Part II solutions

1. The following figure represents a rectangular piece of paper $ABCD$ whose dimensions are 4 inches by 3 inches. When the paper is folded along the line segment $\overline{EF}$, the corners A and C coincide.

   (a) Find the length of segment $\overline{EF}$.

   (b) Extend $\overline{AD}$ and $\overline{EF}$ so they meet at G. Find the area of the triangle $\triangle AEG$.



Solution:

(a)
Since vertices A and C coincide when the paper is folded, $|\overline{AF}| = |\overline{CF}|$ and $|\overline{AE}| = |\overline{CE}|$, hence $\overline{EF}$ is a perpendicular bisector of $\overline{AC}$. Since $\triangle AFC \equiv \triangle AEC$, we have $|\overline{CF}| = |\overline{CE}|$, so that $\overline{AC}$ is a perpendicular bisector of $\overline{EF}$.

Apply the Pythagorean Theorem to $\triangle ADF$:

$$3^2 + (4 - |\overline{AF}|)^2 = |\overline{AF}|^2 \quad \text{yields} \quad |\overline{AF}| = \frac{25}{8}$$

Let X be the intersection of $\overline{AC}$ and $\overline{EF}$. Apply the Pythagorean Theorem to $\triangle AXF$:

$$\left(\frac{|\overline{EF}|}{2}\right)^2 + \left(\frac{5}{2}\right)^2 = \left(\frac{25}{8}\right)^2 \quad \text{yields} \quad |\overline{EF}| = \frac{15}{4}$$

(b)
By symmetry, the area of quadrilateral $AEFD$ is 6 and by (a), $|\overline{DF}| = 7/8$.

Since triangles $\triangle AEG$ and $\triangle DFG$ are similar:

$$\frac{3 + |\overline{DG}|}{|\overline{DG}|} = \frac{25/8}{7/8} \quad \text{yields} \quad |\overline{DG}| = \frac{7}{6}$$

So the area of $\triangle AEG$ is

$$6 + \frac{1}{2} \cdot \frac{7}{8} \cdot \frac{7}{6} = \frac{625}{96}$$

2. (a) Let p be a prime number. If a, b, c, and d are distinct integers such that the equation

$$(x-a)(x-b)(x-c)(x-d)-p^2=0$$

has an integer solution r, show that

$$(r-a)+(r-b)+(r-c)+(r-d)=0.$$

(b) Show that r must be a double root of the equation $(x-a)(x-b)(x-c)(x-d)-p^2=0$.

Solution:

(a)
Since r is a solution to the above equation, substituting r into the equation we see that

$$(r-a)(r-b)(r-c)(r-d)=p^2$$

Since a, b, c, and d are distinct integers, $(r-a)$, $(r-b)$, $(r-c)$, $(r-d)$ are distinct integers whose product is p^2. Since p is prime, the only set of four distinct integers whose product is p^2 is $\{1,-1,p,-p\}$. Thus

$$(r-a)+(r-b)+(r-c)+(r-d)=1-1+p-p=0$$

as desired.

(b)
From (a), $\{(r-a),(r-b),(r-c),(r-d)\}=\{-1,1,-p,p\}$. Thus the left-hand side of the original equation can be written as

$$[(x-r)+(r-a)][(x-r)+(r-b)][(x-r)+(r-c)][(x-r)+(r-d)]-p^2$$

$$[(x-r)-1][(x-r)+1][(x-r)-p][(x-r)+p]-p^2$$

$$[(x-r)^2-1][(x-r)^2-p^2]$$

Therefore

$$[(x-r)^2-1][(x-r)^2-p^2]=p^2$$

$$(x-r)^4-(1+p^2)(x-r)^2=0$$

$$(x-r)^2[(x-r)^2-(1+p^2)]=0$$

and r is a double root.

3. If $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 0$, prove the following statements.

(a) $\cos(x - y) = -\frac{1}{2}$

(b) $\cos(\theta - x) + \cos(\theta - y) + \cos(\theta - z) = 0$, for any angle θ.

(c) $\sin^2 x + \sin^2 y + \sin^2 z = \frac{3}{2}$

Solution:

(a)
From the assumption, $\cos x + \cos y = -\cos z$ and $\sin x + \sin y = -\sin z$. Squaring the two equations, the result follows from adding the resulting equations and simplifying using trigonometric identities $\sin^2 A + \cos^2 A = 1$ and $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

(b)
$\cos(\theta - x) + cos(\theta - y) + cos(\theta - z)$
$= \cos\theta \cos x + \sin\theta \sin x + \cos\theta \cos y + \sin\theta \sin y + \cos\theta \cos z + \sin\theta \sin z$
$= \cos\theta[\cos x + \cos y + \cos z] + \sin\theta[\sin x + \sin y + \sin z]$
$= 0$

(c)
Setting $\theta = x + y + z$ in (b), we get $\cos(x + y) + \cos(y + z) + \cos(x + z) = 0$. Therefore

$$\cos x \cos y + \cos y \cos z + \cos x \cos z = \sin x \sin y + \sin y \sin z + \sin x \sin z$$

Squaring the two equations $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 0$ yields

$$\cos^2 x + \cos^2 y + \cos^2 z + 2(\cos x \cos y + \cos y \cos z + \cos x \cos z) = 0$$

and

$$\sin^2 x + \sin^2 y + \sin^2 z + 2(\sin x \sin y + \sin y \sin z + \sin x \sin z) = 0$$

Therefore

$$\sin^2 x + \sin^2 y + \sin^2 z = \cos^2 x + \cos^2 y + \cos^2 z$$

and

$$2(\sin^2 x + \sin^2 y + \sin^2 z) = \cos^2 x + \cos^2 y + \cos^2 z + \sin^2 x + \sin^2 y + \sin^2 z = 3$$

so

$$\sin^2 x + \sin^2 y + \sin^2 z = \frac{3}{2}$$

4. Let $|A|$ denote the number of elements in the set A.

   (a) Construct an infinite collection $\{A_i\}$ of infinite subsets of the set of natural numbers such that $|A_i \cap A_j| = 0$ for $i \neq j$.

   (b) Construct an infinite collection $\{B_i\}$ of infinite subsets of the set of natural numbers such that $|B_i \cap B_j|$ gives a distinct integer for every pair of i and j, $i \neq j$.

Solution:

(a)
For every prime number p, let

$$A_p = \{p, p^2, p^3, \ldots\}$$

It now follows from unique factorization that $|A_i \cap A_j| = 0$ for $i \neq j$.

(b)
Let P be the set of prime numbers. For every prime number p, let

$$B_p = \{p^i q^j \ : \ p \neq q \in P, \ 1 \leq i \leq p, \ 1 \leq j \leq q\}.$$

Then for $r, s \in P$.

$$B_r \cap B_s = \{r^i s^j \ : \ 1 \leq i \leq r, \ 1 \leq j \leq s\}.$$

Hence $|B_r \cap B_s| = rs$. It now follows from unique factorization that $|B_r \cap B_s|$ gives a distinct integer for every pair of r and s $(r \neq s)$.

5. Consider the equation $x^4 + y^4 = z^5$.

   (a) Show that the equation has a solution where x, y, and z are positive integers.

   (b) Show that the equation has infinitely many solutions where x, y, and z are positive integers.

Solution:

(a)
$x = y = z = 2$ is a solution.

(b)
Set $x = y = 2a$ and $z = 2b$. Then we want $(2a)^4 + (2a)^4 = (2b)^5$. Hence we want $a^4 = b^5$. So set $a = 2^{5i}$ and $b = 2^{4i}$ for $i = 0, 1, 2, \ldots$.