1. Consider the equation

\[ x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_{n-1}x_n + x_nx_1 = 0 \]

where \( x_i \in \{1, -1\} \) for \( i = 1, 2, \ldots, n \).

(a) Show that if the equation has a solution, then \( n \) is even. \( (2 \text{ points}) \)

(b) Suppose \( n \) is divisible by 4. Show that the equation has a solution. \( (2 \text{ points}) \)

(c) Show that if the equation has a solution, then \( n \) is divisible by 4. \( (6 \text{ points}) \)

(a) There are \( n \) summands, each with value 1 or \(-1\). Since the sum is 0, exactly half of them have value 1. Hence \( n \) is even.

(b) Set \( x_1 = 1, x_2 = -1, x_3 = -1, x_4 = 1, x_5 = 1, x_6 = -1, x_7 = -1, x_8 = 1, \ldots \).

In other words, set \( x_{4i+1} = x_{4i+4} = 1 \) and \( x_{4i+2} = x_{4i+3} = -1 \).

It is easy to see that this is a valid solution.

(c) \textit{Solution 1:}

From part (a), we know that \( n \) is even. Let \( n = 2k \). Also from the solution in part (a), we see that exactly \( k \) of the summands are \(-1\). Hence

\[ (x_1x_2)(x_2x_3)(x_3x_4)\cdots(x_{n-1}x_n)(x_nx_1) = (-1)^k \]

On the other hand, the product is also equal to \( x_1^2x_2^2\cdots x_n^2 = 1 \). Hence \( k \) is even and we are done.

\textit{Solution 2:}

Suppose we have a solution \( x_i \) for \( i = 1, 2, \ldots, n \). Suppose \( x_p = -1 \). Consider the effect of changing \( x_p = 1 \). If \( x_{p-1} \neq x_{p+1} \), then such a change will not affect the right-hand side. Otherwise, the right-hand side increases by 4 or decreases by 4. Hence, by changing every \(-1\) to a \( 1 \), the right-hand side will be a multiple of 4. But now the left-hand side is \( n \) and we are done.
2.  
(a) Find a polynomial \( f(x) \) with integer coefficients and two distinct integers \( a \) and \( b \) such that \( f(a) = b \) and \( f(b) = a \). (2 points)

(b) Let \( f(x) \) be a polynomial with integer coefficients and \( a, b, \) and \( c \) be three integers. Suppose \( f(a) = b, \ f(b) = c, \) and \( f(c) = a \). Show that \( a = b = c \). (8 points)

(a) There are many solutions. For example, \( f(x) = 1 - x \) with \( a = 1 \) and \( b = 0 \) will do.

(b) Suppose

\[
 f(x) = d_n x^n + d_{n-1} x^{n-1} + \cdots + d_1 x + d_0.
\]

Suppose \( f(a) = b, \ f(b) = c, \) and \( f(c) = a \). We may assume \( a \leq b \leq c \). If \( b = a \), then \( f(b) = b, \ f(b) = c, \ f(c) = a \). This gives \( b = c \) since \( f \) is a function. Hence \( a = b = c \) and we are done.

Therefore, we may assume \( b \neq a \). Similarly, we may assume \( b \neq c \). Hence, we have \( a < b < c \).

Now consider \( f(a) - f(c) = b - a \). Since

\[
 f(a) - f(c) = d_n(a^n - c^n) + d_{n-1}(a^{n-1} - c^{n-1}) + \cdots + d_1(a - c)
\]

\((a - c)|(f(a) - f(c))\). Hence \((c - a)|(b - a)\). This is impossible as \( b - a \neq 0 \) and \( c - a > b - a \) and we are done.
3. (a) Consider the triangle with vertices $M(0, 2n + 1)$, $S(1, 0)$, and $U(0, \frac{1}{2n^2})$, where $n$ is a positive integer. If $\theta = \angle MSU$, prove that $\tan \theta = 2n - 1$. (5 points)

(b) Find positive integers $a$ and $b$ that satisfy the following equation. (2 points)

$$\arctan \frac{1}{8} = \arctan a - \arctan b$$

(c) Determine the exact value of the following infinite sum. (3 points)

$$\arctan \frac{1}{2} + \arctan \frac{1}{8} + \arctan \frac{1}{18} + \arctan \frac{1}{32} + \cdots + \arctan \frac{1}{2n^2} + \cdots$$

(a) One way to prove this theorem is to let $A$ be the origin, $\alpha = \angle MSA$ and $\beta = \angle USA$. Then $\theta = \alpha - \beta$ and

$$\tan \theta = \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

Since $\tan \alpha = 2n + 1$ and $\tan \beta = \frac{1}{2n^2}$, we have

$$\tan \theta = \frac{4n^3 + 2n^2 - 1}{2n^2 + 2n + 1} = 2n - 1$$

(b) Let $n = 2$, then $\tan \alpha = 5$, $\tan \beta = \frac{1}{8}$, and $\tan \theta = 3$. Since $\beta = \alpha - \theta$, we have

$$\tan^{-1} \frac{1}{8} = \tan^{-1} 5 - \tan^{-1} 3$$

(c) The key is to observe that

$$\tan^{-1} \frac{1}{2} = \tan^{-1} 3 - \tan^{-1} 1$$
$$\tan^{-1} \frac{1}{8} = \tan^{-1} 5 - \tan^{-1} 3$$
$$\tan^{-1} \frac{1}{18} = \tan^{-1} 7 - \tan^{-1} 5$$

and, in general

$$\tan^{-1} \frac{1}{2n^2} = \tan^{-1}(2n + 1) - \tan^{-1}(2n - 1)$$

So a partial sum looks like

$$(\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 5 - \tan^{-1} 3) + \cdots + (\tan^{-1}(2n + 1) - \tan^{-1}(2n - 1)) = \tan^{-1}(2n + 1) - \tan^{-1} 1$$

which approaches $\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ as $n$ increases.
4. (a) Prove: \((55 + 12\sqrt{21})^{1/3} + (55 - 12\sqrt{21})^{1/3} = 5\). (5 points)

(b) Completely factor \(x^8 + x^6 + x^4 + x^2 + 1\) into polynomials with integer coefficients, and explain why your factorization is complete. (5 points)

(a)
Let \(a = 55\) and \(b = 12\sqrt{21}\). Note that \(a^2 - b^2 = 1\). Let \(x = (a + b)^{1/3} + (a - b)^{1/3}\). Cubing this equation and simplifying gives us

\[x^3 = 2a + 3(a + b)^{1/3}(a - b)^{1/3} \left( (a + b)^{1/3} + (a - b)^{1/3} \right)\]

This reduces to \(x^3 = 2a + 3x\), so the problem now is to solve \(x^3 - 3x - 110 = 0\). The cubic factors into \((x - 5)(x^2 + 5x + 22)\). The only solution is \(x = 5\).

(b)
\[x^8 + x^6 + x^4 + x^2 + 1 = \frac{x^{10} - 1}{x^2 - 1} = \frac{(x^5 - 1)(x^5 + 1)}{x^2 - 1} = \frac{(x - 1)(x^4 + x^3 + x^2 + x + 1)(x + 1)(x^4 - x^3 + x^2 - x + 1)}{(x - 1)(x + 1)} = (x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1)\]

To show the factorization is complete, we only need to show \(f(x) = x^4 + x^3 + x^2 + x + 1\) is irreducible, since the other one is \(f(-x)\).

Since \(f(x)\) has no real zeros, it doesn’t have any linear factors. Without loss of generality, suppose \(x^2 + bx + c\) is a factor, necessarily, \(c = \pm 1\).

Since \(b^2 - 4c < 0\), \(c\) must be +1 and \(b\) could only be -1, 0, or +1, leaving three possible candidates:

\[p(x) = x^2 - x + 1 \quad q(x) = x^2 + x + 1 \quad r(x) = x^2 + 1\]

Since \(p(-1) = 3\), which is not a factor of \(f(-1) = 1\), \(p(x)\) is not a factor of \(f(x)\).

Similarly, \(q(1) = 3\) and \(r(1) = 2\), neither is a factor of \(f(1) = 5\), eliminating \(q(x)\) and \(r(x)\).
5. In this problem, we simulate a hula hoop as it gyrates about your waist. We model this situation by representing the hoop with a rotating a circle of radius 2 initially centered at \((-1, 0)\), and representing your waist with a fixed circle of radius 1 centered at the origin. Suppose we mark the point on the hoop that initially touches the fixed circle with a black dot (see the left figure).

As the circle of radius 2 rotates, this dot will trace out a curve in the plane (see the right figure). Let \(\theta\) be the angle between the positive \(x\)-axis and the ray that starts at the origin and goes through the point where the fixed circle and circle of radius 2 touch. Determine formulas for the coordinates of the position of the dot, as functions \(x(\theta)\) and \(y(\theta)\). The left figure shows the situation when \(\theta = 0\) and the right figure shows the situation when \(\theta = \frac{2\pi}{3}\). (10 points)

Let \(O\) be the origin \((0, 0)\), \(C\) be the position of the center of the hoop, and \(D\) be the position of the dot.

Consider the vectors \(\overrightarrow{OC}\) and \(\overrightarrow{CD}\).

The length of \(\overrightarrow{OC}\) is 1 and its argument is \(\theta + \pi\).

The length of \(\overrightarrow{CD}\) is 2 and its argument is \(\frac{\theta}{2}\).

Since \(\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}\)

\[
\overrightarrow{OD} = (1 \cos(\theta + \pi), 1 \sin(\theta + \pi)) + \left(2 \cos \left(\frac{\theta}{2}\right), 2 \sin \left(\frac{\theta}{2}\right)\right)
\]

\[
= (- \cos(\theta), - \sin(\theta)) + \left(2 \cos \left(\frac{\theta}{2}\right), 2 \sin \left(\frac{\theta}{2}\right)\right)
\]

\[
= \left(2 \cos \left(\frac{\theta}{2}\right) - \cos(\theta), 2 \sin \left(\frac{\theta}{2}\right) - \sin(\theta)\right)
\]

Therefore, in terms of \(\theta\) the \(x\)- and \(y\)-coordinates of the dot are

\[
x(\theta) = 2 \cos \left(\frac{\theta}{2}\right) - \cos(\theta) \quad y(\theta) = 2 \sin \left(\frac{\theta}{2}\right) - \sin(\theta)
\]