FORTY-SEVENTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Part II solutions

1. Consider the equation

$$x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_{n-1}x_n + x_nx_1 = 0$$

where $x_i \in \{1, -1\}$ for $i = 1, 2, \ldots, n$.

(a) Show that if the equation has a solution, then n is even. (*2 points*)

(b) Suppose n is divisible by 4. Show that the equation has a solution. (*2 points*)

(c) Show that if the equation has a solution, then n is divisible by 4. (*6 points*)

(a)

There are n summands, each with value 1 or -1. Since the sum is 0, exactly half of them have value 1. Hence n is even.

(b)

Set $x_1 = 1$, $x_2 = -1$, $x_3 = -1$, $x_4 = 1$, $x_5 = 1$, $x_6 = -1$, $x_7 = -1$, $x_8 = 1$, $\ldots$.

In other words, set $x_{4i+1} = x_{4i+4} = 1$ and $x_{4i+2} = x_{4i+3} = -1$.

It is easy to see that this is a valid solution.

(c)

Solution 1:

From part (a), we know that n is even. Let $n = 2k$. Also from the solution in part (a), we see that exactly k of the summands are -1. Hence

$$(x_1x_2)(x_2x_3)(x_3x_4)\cdots(x_{n-1}x_n)(x_nx_1) = (-1)^k$$

On the other hand, the product is also equal to $x_1^2x_2^2\cdots x_n^2 = 1$. Hence k is even and we are done.

Solution 2:

Suppose we have a solution x_i for $i = 1, 2, \ldots, n$. Suppose $x_p = -1$. Consider the effect of changing $x_p = 1$. If $x_{p-1} \neq x_{p+1}$, then such a change will not affect the right-hand side. Otherwise, the right-hand side increases by 4 or decreases by 4. Hence, by changing every -1 to a 1, the right-hand side will be a multiple of 4. But now the left-hand side is n and we are done.

2. (a) Find a polynomial $f(x)$ with integer coefficients and two distinct integers a and b such that $f(a) = b$ and $f(b) = a$. (*2 points*)

   (b) Let $f(x)$ be a polynomial with integer coefficients and a, b, and c be three integers. Suppose $f(a) = b$, $f(b) = c$, and $f(c) = a$. Show that $a = b = c$. (*8 points*)

(a)

There are many solutions. For example, $f(x) = 1 - x$ with $a = 1$ and $b = 0$ will do.

(b)

Suppose

$$f(x) = d_n x^n + d_{n-1}x^{n-1} + \cdots + d_1 x + d_0.$$

Suppose $f(a) = b$, $f(b) = c$ and $f(c) = a$. We may assume $a \leq b \leq c$. If $b = a$, then $f(b) = b$, $f(b) = c$, $f(c) = a$. This gives $b = c$ since f is a function. Hence $a = b = c$ and we are done.

Therefore, we may assume $b \neq a$. Similarly, we may assume $b \neq c$. Hence, we have $a < b < c$.

Now consider $f(a) - f(c) = b - a$. Since

$$f(a) - f(c) = d_n(a^n - c^n) + d_{n-1}(a^{n-1} - c^{n-1}) + \cdots + d_1(a - c)$$

$(a-c) | (f(a) - f(c))$. Hence $(c-a)|(b-a)$. This is impossible as $b - a \neq 0$ and $c - a > b - a$ and we are done.

3. (a) Consider the triangle with vertices $M(0, 2n+1)$, $S(1, 0)$, and $U\left(0, \frac{1}{2n^2}\right)$, where n is a positive integer. If $\theta = \angle MSU$, prove that $\tan\theta = 2n - 1$. (*5 points*)

(b) Find positive integers a and b that satisfy the following equation. (*2 points*)

$$\arctan\frac{1}{8} = \arctan a - \arctan b$$

(c) Determine the exact value of the following infinite sum. (*3 points*)

$$\arctan\frac{1}{2} + \arctan\frac{1}{8} + \arctan\frac{1}{18} + \arctan\frac{1}{32} + \cdots + \arctan\frac{1}{2n^2} + \cdots$$

(a)

One way to prove this theorem is to let A be the origin, $\alpha = \angle MSA$ and $\beta = \angle USA$. Then $\theta = \alpha - \beta$ and

$$\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Since $\tan\alpha = 2n+1$ and $\tan\beta = \dfrac{1}{2n^2}$, we have

$$\tan\theta = \frac{4n^3 + 2n^2 - 1}{2n^2 + 2n + 1} = 2n - 1$$

(b)

Let $n = 2$, then $\tan\alpha = 5$, $\tan\beta = \dfrac{1}{8}$, and $\tan\theta = 3$. Since $\beta = \alpha - \theta$, we have

$$\tan^{-1}\frac{1}{8} = \tan^{-1}5 - \tan^{-1}3$$

(c)

The key is to observe that

$$\tan^{-1}\frac{1}{2} = \tan^{-1}3 - \tan^{-1}1$$
$$\tan^{-1}\frac{1}{8} = \tan^{-1}5 - \tan^{-1}3$$
$$\tan^{-1}\frac{1}{18} = \tan^{-1}7 - \tan^{-1}5$$

and, in general

$$\tan^{-1}\frac{1}{2n^2} = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

So a partial sum looks like

$$(\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) + \cdots + (\tan^{-1}(2n+1) - \tan^{-1}(2n-1)) = \tan^{-1}(2n+1) - \tan^{-1}1$$

which approaches $\dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}$ as n increases.

4. (a) Prove: $\left(55+12\sqrt{21}\right)^{1/3}+\left(55-12\sqrt{21}\right)^{1/3}=5$. (*5 points*)

(b) Completely factor $x^8+x^6+x^4+x^2+1$ into polynomials with integer coefficients, and explain why your factorization is complete. (*5 points*)

(a)

Let $a=55$ and $b=12\sqrt{21}$. Note that $a^2-b^2=1$. Let $x=(a+b)^{1/3}+(a-b)^{1/3}$. Cubing this equation and simplifying gives us

$$x^3=2a+3(a+b)^{1/3}(a-b)^{1/3}\left((a+b)^{1/3}+(a-b)^{1/3}\right)$$

This reduces to $x^3=2a+3x$, so the problem now is to solve $x^3-3x-110=0$.

The cubic factors into $(x-5)(x^2+5x+22)$. The only solution is $x=5$.

(b)

$$\begin{aligned} x^8+x^6+x^4+x^2+1 &= \frac{x^{10}-1}{x^2-1} \\ &= \frac{(x^5-1)(x^5+1)}{x^2-1} \\ &= \frac{(x-1)(x^4+x^3+x^2+x+1)(x+1)(x^4-x^3+x^2-x+1)}{(x-1)(x+1)} \\ &= (x^4+x^3+x^2+x+1)(x^4-x^3+x^2-x+1) \end{aligned}$$

To show the factorization is complete, we only need to show $f(x)=x^4+x^3+x^2+x+1$ is irreducible, since the other one is $f(-x)$.

Since f(x) has no real zeros, it doesn't have any linear factors. Without loss of generality, suppose x^2+bx+c is a factor, necessarily, $c=\pm1$.

Since $b^2-4c<0$, c must be $+1$ and b could only be -1, 0, or $+1$, leaving three possible candidates:

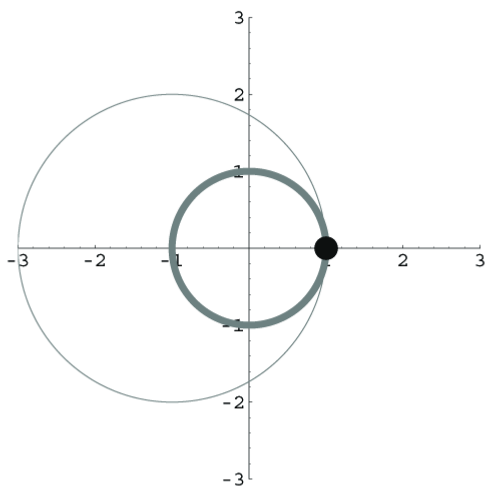
$$p(x)=x^2-x+1 \qquad q(x)=x^2+x+1 \qquad r(x)=x^2+1$$

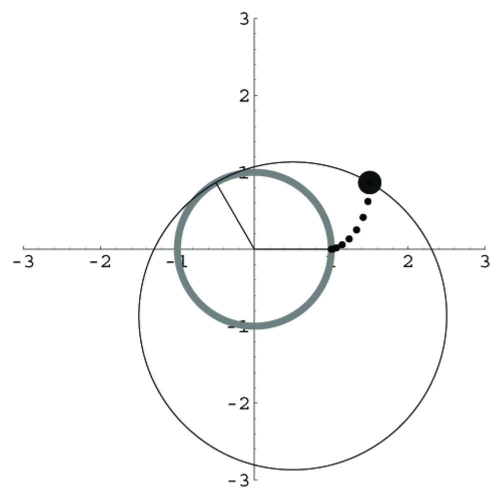
Since $p(-1)=3$, which is not a factor of $f(-1)=1$, $p(x)$ is not a factor of $f(x)$.

Similarly, $q(1)=3$ and $r(1)=2$, neither is a factor of $f(1)=5$, eliminating $q(x)$ and $r(x)$.

5. In this problem, we simulate a hula hoop as it gyrates about your waist. We model this situation by representing the hoop with a rotating a circle of radius 2 initially centered at $(-1, 0)$, and representing your waist with a fixed circle of radius 1 centered at the origin. Suppose we mark the point on the hoop that initially touches the fixed circle with a black dot (see the left figure).

   As the circle of radius 2 rotates, this dot will trace out a curve in the plane (see the right figure). Let θ be the angle between the positive x-axis and the ray that starts at the origin and goes through the point where the fixed circle and circle of radius 2 touch. Determine formulas for the coordinates of the position of the dot, as functions $x(\theta)$ and $y(\theta)$. The left figure shows the situation when $\theta = 0$ and the right figure shows the situation when $\theta = 2\pi/3$. (*10 points*)





   Let O be the origin $(0, 0)$, C be the position of the center of the hoop, and D be the position of the dot.

   Consider the vectors $\overrightarrow{OC}$ and $\overrightarrow{CD}$.

   The length of $\overrightarrow{OC}$ is 1 and its argument is $\theta + \pi$.

   The length of $\overrightarrow{CD}$ is 2 and its argument is $\frac{\theta}{2}$.

   Since $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$$\begin{aligned}
\overrightarrow{OD} &= (1\cos(\theta+\pi), 1\sin(\theta+\pi)) + \left(2\cos\left(\frac{\theta}{2}\right), 2\sin\left(\frac{\theta}{2}\right)\right) \\
&= (-\cos(\theta), -\sin(\theta)) + \left(2\cos\left(\frac{\theta}{2}\right), 2\sin\left(\frac{\theta}{2}\right)\right) \\
&= \left(2\cos\left(\frac{\theta}{2}\right) - \cos(\theta), 2\sin\left(\frac{\theta}{2}\right) - \sin(\theta)\right)
\end{aligned}$$

   Therefore, in terms of θ the x- and y-coordinates of the dot are

$$x(\theta) = 2\cos\left(\frac{\theta}{2}\right) - \cos(\theta) \qquad y(\theta) = 2\sin\left(\frac{\theta}{2}\right) - \sin(\theta)$$