MMPC STUDENT CODE

FORTY-SEVENTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

December 3, 2003

INSTRUCTIONS
(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit you from asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	1	2	3	4	5	Total
Score						

1. Consider the equation

$$x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_{n-1}x_n + x_nx_1 = 0$$

where $x_i \in \{1, -1\}$ for $i = 1, 2, \ldots, n$.

   (a) Show that if the equation has a solution, then n is even. (*2 points*)

   (b) Suppose n is divisible by 4. Show that the equation has a solution. (*2 points*)

   (c) Show that if the equation has a solution, then n is divisible by 4. (*6 points*)

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2. (a) Find a polynomial $f(x)$ with integer coefficients and two distinct integers a and b such that $f(a) = b$ and $f(b) = a$. (*2 points*)

   (b) Let $f(x)$ be a polynomial with integer coefficients and a, b, and c be three integers. Suppose $f(a) = b$, $f(b) = c$, and $f(c) = a$. Show that $a = b = c$. (*8 points*)

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3. (a) Consider the triangle with vertices $M(0, 2n+1)$, $S(1, 0)$, and $U\left(0, \frac{1}{2n^2}\right)$, where n is a positive integer. If $\theta = \angle MSU$, prove that $\tan\theta = 2n - 1$. (*5 points*)

   (b) Find positive integers a and b that satisfy the following equation. (*2 points*)

$$\arctan\frac{1}{8} = \arctan a - \arctan b$$

   (c) Determine the exact value of the following infinite sum. (*3 points*)

$$\arctan\frac{1}{2} + \arctan\frac{1}{8} + \arctan\frac{1}{18} + \arctan\frac{1}{32} + \cdots + \arctan\frac{1}{2n^2} + \cdots$$

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4. (a) Prove: $\left(55 + 12\sqrt{21}\right)^{1/3} + \left(55 - 12\sqrt{21}\right)^{1/3} = 5$. (*5 points*)

   (b) Completely factor $x^8 + x^6 + x^4 + x^2 + 1$ into polynomials with integer coefficients, and explain why your factorization is complete. (*5 points*)

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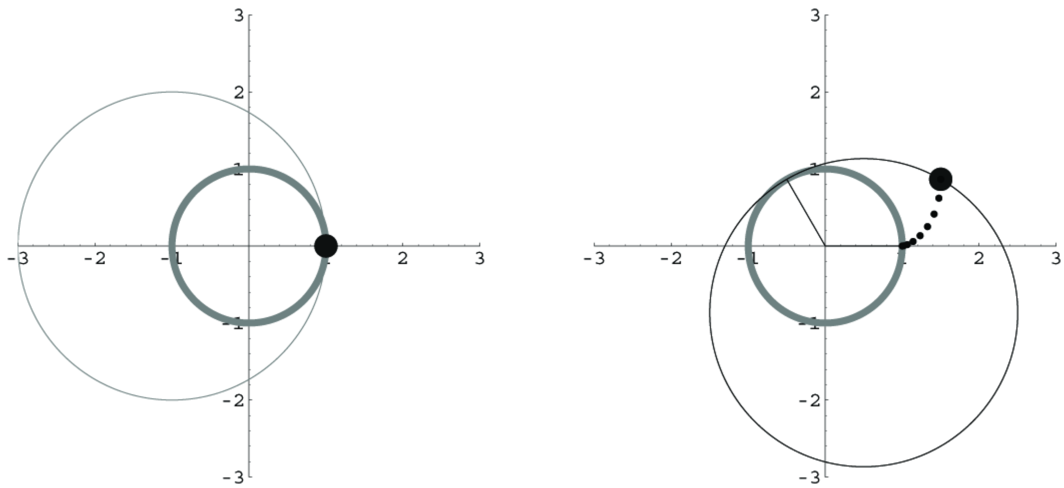
5. In this problem, we simulate a hula hoop as it gyrates about your waist. We model this situation by representing the hoop with a rotating a circle of radius 2 initially centered at $(-1, 0)$, and representing your waist with a fixed circle of radius 1 centered at the origin. Suppose we mark the point on the hoop that initially touches the fixed circle with a black dot (see the left figure).

   As the circle of radius 2 rotates, this dot will trace out a curve in the plane (see the right figure). Let θ be the angle between the positive x-axis and the ray that starts at the origin and goes through the point where the fixed circle and circle of radius 2 touch. Determine formulas for the coordinates of the position of the dot, as functions $x(\theta)$ and $y(\theta)$. The left figure shows the situation when $\theta = 0$ and the right figure shows the situation when $\theta = 2\pi/3$. (*10 points*)



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(continued solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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