FORTY-SEVENTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 8, 2003

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.
2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.
3. Consider the problems and responses carefully. You may work out ideas on scratch paper before selecting a response.
4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.
5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(e) None of the above." If you believe none of the first four alternatives is correct, choose response (e).
6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.
7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.
8. You may now open the test booklet and begin.

1. The expression $2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}$ is equal to

(a) $\frac{157}{68}$ (b) $\frac{121}{60}$ (c) 2.3 (d) $\frac{141}{68}$ (e) None of the above.

2. Consider the ordered data set

$$\{0, 2, 3, 3, 6, 8, 11, 16, 18, 19, 20, 24, 25, 29, 33, 34, 35, 36, 40, 45\}$$

The median of this set is

(a) 19 (b) 19.50 (c) 20 (d) 20.45 (e) 3

3. A certain square table can seat at most one person on each side. If you line up 1000 of these square tables in a single row side by side, how many people can you seat?

(a) 4000 (b) 2000 (c) 2002 (d) 2003 (e) 2004

4. If $\odot x$ is defined to be $x^2 + 3x$, find the value of $\odot \odot 2$.

(a) 10 (b) 20 (c) 120 (d) 100 (e) 130

5. In a right triangle with legs a and b and hypotenuse c, the angles A, B, and C are opposite to the sides a, b, and c respectively. What does $e^{\ln b - \ln a}$ equal?

(a) $\sin A$ (b) $\cos B$ (c) $\tan A$ (d) $\cot A$ (e) $\csc A$

6. A man born in the first half of the nineteenth century was x years old in the year x^2. In what year was he born?

(a) 1849 (b) 1825 (c) 1812 (d) 1836 (e) 1806

7. A zoo has a collection of birds and dogs. The animals in this collection have a total of 44 eyes and 72 feet. How many dogs are in this collection?

(a) 8 (b) 14 (c) 15 (d) 16 (e) None of the above.

8. How many times does the digit 1 appear in the list of whole numbers from 0 to 100 inclusive?

(a) 18 times (b) 20 times (c) 22 times (d) 24 times (e) None of the above.

9. One-half of the sum of two consecutive even numbers is 73. What is the product of these two consecutive even numbers?

(a) 5328 (b) 7274 (c) 5382 (d) 7220 (e) 7472

10. In a given right triangle the degree measure of one of the angles is the smallest prime number whose square is greater than 1700, and the side opposite that angle has length 1. Which of the following is closest to the length of the hypotenuse of this triangle?

(a) 1.072 (b) 1.367 (c) 1.202 (d) 1.466 (e) 1.524

11. The last digit of 7^{2003} is

(a) 1 (b) 3 (c) 5 (d) 7 (e) 9

12. Consider a regular polygon with 2003 sides. The distance from the center of the polygon to a vertex is 5 units. Which of the following is closest to the area of the polygon?

(a) 1 (b) 31 (c) 49 (d) 78 (e) 2003

13. The value of following the following quotient is an integer. Find the sum of its digits.

$$\frac{12345678901234567890}{(12345678901234567891)^2 - (12345678901234567890)(12345678901234567892)}$$

(a) 85 (b) 90 (c) 95 (d) 100 (e) None of the above.

14. In three dimensional space, consider the point $(x, y, z) = (4, 3, 1)$. Rotate this point 45 degrees counter-clockwise about the z-axis (as viewed from above). What are the coordinates of the new point?

(a) $\left(\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}, 1\right)$

(b) $\left(\frac{4\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(c) $\left(\frac{7\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1\right)$

(d) $\left(\frac{3\sqrt{2}}{2}, \frac{4\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(e) $(1, 7, 1)$

15. Which of the following is equal to $\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right)\cdots\left(1 - \frac{1}{n}\right)$?

(a) $\frac{1}{n}$ (b) $\frac{2(n-1)}{n}$ (c) $\frac{2}{n(n+1)}$ (d) $\frac{2}{n}$ (e) $\frac{3}{n(n+1)}$

16. The sequence $a_1 = 2$, $a_2 = 3$, $a_3 = 8$, $a_4 = 19$, $a_5 = 46$ satisfies which of the following recurrence relations?

(a) $a_{n+1} = a_n + 4n^2 - 8n + 5$
(b) $a_{n+1} = a_n + 4n^2 + 1$
(c) $a_{n+2} = a_{n+1} + a_n$
(d) $a_{n+2} = 2a_{n+1} + a_n$
(e) None of the above.

17. When 600 students are seated in m rows, each of which contains n chairs, every chair is occupied. If five more chairs were in each row, everyone could be seated in four fewer rows and again every chair would be occupied. What is the value of $m + n$?

(a) 49 (b) 48 (c) 47 (d) 46 (e) 45

18. If x is a real number, then the inequality $1 \leq |x - 2| \leq 7$ is equivalent to which of the following statements?

(a) $x \leq 1$ or $x \geq 3$
(b) $1 \leq x \leq 3$
(c) $-5 \leq x \leq 1$ or $3 \leq x \leq 9$
(d) $-6 \leq x \leq 1$ or $3 \leq x \leq 10$
(e) $-5 \leq x \leq 9$

19. If the positive y-axis indicates the direction "north" and you translate every point on the graph of $y = (x - 3)(x - 7)$ a distance of 2 units in the northeast direction, then the smallest y-value on the new graph occurs at which of the following points?

(a) $(7, 0)$
(b) $(7, -2)$
(c) $\left(5 - \sqrt{2}, 4 + \sqrt{2}\right)$
(d) $\left(5 + \sqrt{2}, -4 + \sqrt{2}\right)$
(e) None of the above.

20. An analog clock has an hour hand, a minute hand, and a second hand. In a 24-hour day (12:00:00 a.m. to 11:59:59 p.m.), how many times will a hand point to the 12? (Note that at 12:00:00, all three hands point to 12, contributing 3 to the total.)

(a) 1462 (b) 1464 (c) 1466 (d) 1468 (e) 1470

21. How many integers from 1 to 2003 inclusive have an odd number of distinct positive divisors?

(a) 100 (b) 75 (c) 50 (d) 46 (e) 44

22. If a is a positive integer, how many solutions are there to the equation $\sin ax + \cos ax = 1$ on the interval $0 \leq x < \dfrac{2\pi}{a}$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 8

23. What is the area of the region in the first quadrant bounded by the x-axis, the line $y = 2x$, and the circles $x^2 + y^2 = 20$ and $x^2 + y^2 = 30$?

(a) $5 \arctan 2$ (b) $\dfrac{5\pi}{4}$ (c) $\dfrac{5\pi}{2}$ (d) 5 (e) None of the above.

24. How many squares, of any size, are formed on a standard chessboard by its individual squares? (see *figure 1*)

(a) 200
(b) 202
(c) 204
(d) 206
(e) None of the above.

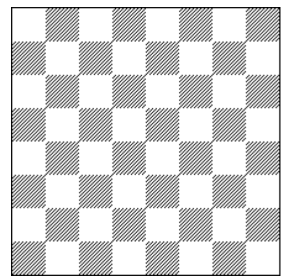

figure 1

25. In a sequence of positive numbers, each term except the first two is the sum of all its predecessors. The eleventh term of the sequence is 1000 and the first term is 1. What is the second term?

(a) $\dfrac{93}{32}$ (b) $\dfrac{32}{93}$ (c) $\dfrac{32}{23}$ (d) $\dfrac{23}{32}$ (e) None of the above.

26. Which of the following is equal to the set of all real numbers b such that $x^2 + bx + 5 = 0$ has two distinct real solutions?

(a) $\{b \,|\, b > 4\}$
(b) $\{b \,|\, b < -4\}$
(c) $\{b \,|\, b > 4\} \cup \{b \,|\, b < -4\}$
(d) $\left\{b \,|\, b > 2\sqrt{5}\right\} \cup \left\{b \,|\, b < -2\sqrt{5}\right\}$
(e) $\left\{b \,|\, b > 2\sqrt{5}\right\}$

27. If $f(x) = \sum_{k=3}^{+\infty} \sin^k(x)$, what does $f\left(\dfrac{\pi}{6}\right)$ equal?

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{3\sqrt{3}}{8 - 4\sqrt{3}}$ (e) $+\infty$

28. Two swimmers, at opposite ends of a 90-foot pool, start to swim the length of the pool, one at the rate of 3 feet per second, the other at 2 feet per second. They swim back and forth for 12 minutes. Allowing no loss of time at the turns, what is the number of times they meet each other?

(a) 24 (b) 21 (c) 20 (d) 19 (e) 18

29. If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number will be divisible by 11?

(a) $\frac{2}{5}$ (b) $\frac{1}{5}$ (c) $\frac{1}{10}$ (d) $\frac{1}{11}$ (e) $\frac{1}{15}$

30. For $a > 1$, the solution set of the equation $\log_a x + \log_a(x - 2a) = 2$ is

(a) $\left\{a + a\sqrt{2}\right\}$

(b) $\left\{a - a\sqrt{2}\right\}$

(c) $\left\{a\sqrt{2} - a\right\}$

(d) $\left\{a + a\sqrt{2}, a - a\sqrt{2}\right\}$

(e) $\left\{a\sqrt{2} + a, a\sqrt{2} - a\right\}$

31. If p is the perimeter of an equilateral triangle inscribed in a circle, what is the area of the circle?

(a) $\frac{\pi p^2}{3}$ (b) $\frac{\pi p^2}{9}$ (c) $\frac{\pi p^2}{27}$ (d) $\frac{\pi p^2}{81}$ (e) $\frac{\pi p^2\sqrt{3}}{27}$

32. Triangle $\triangle ABC$ has a right angle at A. Let $\overline{AD}$ be the altitude from A to the base $\overline{BC}$. The length of $\overline{BD}$ is 6 and the length of $\overline{CD}$ is 24. Find the length of $\overline{AD}$.

(a) 10 (b) $6\sqrt{5}$ (c) $12\sqrt{5}$ (d) $2\sqrt{13}$ (e) 12

33. Which of the following is an equation of a line that is parallel to, and a distance of four units from, the line $y = \frac{3}{4}x + 6$?

(a) $y = \frac{3}{4}x + 1$ (b) $y = \frac{3}{4}x - 1$ (c) $y = \frac{3}{4}x$ (d) $y = \frac{3}{4}x + 2$ (e) $y = \frac{3}{4}x - \frac{2}{3}$

34. If a and b are non-zero real numbers, then the function $f(x) = \dfrac{abx}{a^2 + x^2}$ is

(a) always positive.
(b) undefined at $x = \pm a$.
(c) equal to $\arctan(abx)$.
(d) equal to 0 for exactly three values of x.
(e) an odd function.

35. Let $s_n = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$. As n gets larger, the difference of s_n and which of these numbers diminishes towards zero?

(a) π (b) 3 (c) $\dfrac{69}{25}$ (d) e (e) $\dfrac{\pi\sqrt{3}}{2}$

36. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$ for $-1 < x < 1$, then $f\left(\dfrac{3x + x^3}{1 + 3x^2}\right)$ is equal to which of the following?

(a) $-f(x)$ (b) $2f(x)$ (c) $3f(x)$ (d) $(f(x))^3$ (e) $(f(x))^3 - f(x)$

37. If a cube is inscribed in a sphere of diameter 3, then the volume of the cube is

(a) 1 (b) $\sqrt{3}$ (c) 3 (d) 27 (e) $3\sqrt{3}$

38. Square $ABCD$ has sides of length 14 and a circle is drawn through A and D so that it is tangent to $\overline{BC}$ (see *figure 2*). What is the radius of the circle?

(a) 8.5
(b) 8.75
(c) 9
(d) 9.25
(e) 9.5

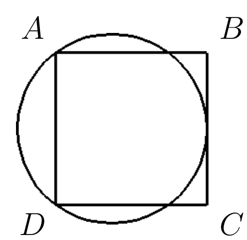

figure 2

39. How many 5-digit numbers are there satisfying the condition that every digit is greater than the digit to its right?

(a) 248 (b) 250 (c) 252 (d) 254 (e) 256

40. The sides of a six-sided die are labeled as follows: 0, 0, 1, -1, i, $-i$, where $i = \sqrt{-1}$. If you roll two of these dice, what is the probability that the sum is 0?

(a) $\dfrac{1}{12}$ (b) $\dfrac{1}{9}$ (c) $\dfrac{1}{6}$ (d) $\dfrac{2}{9}$ (e) $\dfrac{1}{3}$

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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