FORTY-SIXTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Part II solutions

1. (a) Show that for every fixed positive integer $m > 1$, there are always positive integers x and y such that $x^2 - y^2 = m^3$.

   (b) Find all of the positive integer solutions of the equation $x^6 = y^2 + 127$.

Solution

(a)

After observing:

m	m^3	$x^2 - y^2$
2	8	$3^2 - 1^2$
3	27	$6^2 - 3^2$
4	64	$10^2 - 6^2$
5	125	$15^2 - 10^2$

we speculate that

$$m^3 = (1 + \cdots + m - 1 + m)^2 - (1 + \cdots + m - 1)^2$$

Factoring the right-hand side of this equation as the difference of two squares yields

$$(1 + \cdots + m - 1 + m)^2 - (1 + \cdots + m - 1)^2 = m(m + \cdots + m) = m(mm) = m^3$$

(b)

Assume x and y are positive integers and $x^6 = y^2 + 127$.

Since $x^6 - y^2 = 127$, we factor the left-hand side as the difference of two squares and write $(x^3 - y)(x^3 + y) = 127$.

Since 127 is prime, one of these factors must equal 127 and the other must equal 1. It must be the case that $x^3 - y = 1$ and $x^3 + y = 127$.

Since $x^3 = y + 1$, substitution yields $2y + 1 = 127$, then $y = 63$ and $x = 4$.

This is the only solution to $x^6 = y^2 + 127$ if we require x and y to be positive integers.

2. (a) Let $P(x)$ be a polynomial with integer coefficients. Suppose that $P(0)$ is an odd integer and that $P(1)$ is also an odd integer. Show that if c is an integer then $P(c)$ is not equal to 0. (*5 points*)

(b) Let $P(x)$ be a polynomial with integer coefficients. Suppose that $P(1{,}000) = 1{,}000$ and $P(2{,}000) = 2{,}000$. Explain why $P(3{,}000)$ cannot be equal to 1,000. (*5 points*)

Solution

(a)

Let c be an integer. To show $P(c) \neq 0$, we show something stronger, namely, $P(c)$ is odd.

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Since $P(0)$ and $P(1)$ are odd, a_0 is odd and $a_n + a_{n-1} + \cdots + a_1 + a_0$ is odd. Hence $a_n + a_{n-1} + \cdots + a_1$ is even. Therefore, the number of odd integers in $a_n, a_{n-1}, \ldots, a_1$ is even.

Now $P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$. Recall that our goal is to show that $P(c)$ is odd. This claim is clearly true if c is even as a_0 is odd. Hence we may now assume c to be odd. Recall that the number of odd integers in $a_n, a_{n-1}, \ldots, a_1$ is even. This implies $a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c$ is even and $a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 = P(c)$ is odd, so we are done.

(b)

Consider $Q(x)$ defined to be $P(x + 1000) - 1000$. Then the problem can be restated as:

> Suppose $Q(0) = 0$ and $Q(1000) = 1000$; show that $Q(2000) \neq 0$.

Let $Q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then $Q(0) = 0$ implies $a_0 = 0$. Therefore $Q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$. Hence

$$Q(x) = x(a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1)$$

Let $f(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1$. Now $Q(1000) = 1000$ implies $f(1000) = 1$, that is, $a_n(1000)^{n-1} + a_{n-1}(1000)^{n-2} + \cdots + a_2(1000) + a_1 = 1$. Hence a_1 is odd.

Now $Q(2000) = 2000 f(2000)$ and

$$f(2000) = a_n(2000)^{n-1} + a_{n-1}(2000)^{n-2} + \cdots + a_2(2000) + a_1.$$

Since a_1 is odd, $f(2000)$ is odd. Therefore $f(2000) \neq 0$ which implies $Q(2000) \neq 0$.

3. Triangle $\triangle ABC$ is created from points $A(0,0)$, $B(1,0)$ and $C(1/2,2)$. Let q, r, and s be numbers such that $0 < q < 1/2 < s < 1$, and $q < r < s$. Let D be the point on AC which has x-coordinate q, E be the point on AB which has x-coordinate r, and F be the point on BC that has x-coordinate s.

   (a) Find the area of triangle $\triangle DEF$ in terms of q, r, and s. (*6 points*)

   (b) If $r = 1/2$, prove that at least one of the triangles $\triangle ADE$, $\triangle CDF$, or $\triangle BEF$ has an area of at least 1/4. (*4 points*)

Solution

(a)

Solution #1:

The area of triangle $\triangle DEF$ is the area of triangle $\triangle ABC$, less the areas of triangles $\triangle ADE$, $\triangle BEF$, and $\triangle CDF$. The area of triangle $\triangle ABC$ is 1. The area of triangle $\triangle ADE$ is $2rq$. The area of triangle $\triangle BEF$ is $2(1-r)(1-s)$.

The area of triangle $\triangle CDF$ is $(1-2q)(2s-1)$ (To get this area, use $(1/2)|CD||CF|\sin\angle DCF$). Combining these expressions together, and simplifying, produces the area of triangle $\triangle DEF$ as $-2q + 2r - 2qr - 2rs + 4qs$.

Solution #2:

Create a rectangle by dropping a perpendiculars to AB through D and F, and then drawing a line parallel to AB through D. AB includes the fourth side. Then, the area of triangle $\triangle DEF$ is the area of the rectangle less the areas of the three new triangles that are created. The area of the rectangle is $4q(s-q)$, while the three new triangles have areas $2q(r-q)$, $2(1-s)(s-r)$, and $2(s-q)(s+q-1)$. Combining these expressions together, and simplifying, produces the area of triangle $\triangle DEF$ as $-2q + 2r - 2qr - 2rs + 4qs$.

(b)

Using solution #1 above, and substituting $r = 1/2$, we get that the area of the three triangles is q, $1-s$, and $(1-2q)(2s-1)$. Assumethe areas of the first two triangles are less than 1/4, then $q < 1/4$ and $s > 3/4$. However, then $1-2q > 1/2$, and $2s-1 > 1/2$, so the third triangle would have area greater than 1/4.

4. In the Gregorian calendar:

   (i) years not divisible by 4 are common years,
   (ii) years divisible by 4 but not by 100 are leap years,
   (iii) years divisible by 100 but not by 400 are common years,
   (iv) years divisible by 400 are leap years,
   (v) a leap year contains 366 days; a common year 365 days.

   From the information above:

   (a) Find the number of common years and leap years in 400 consecutive Gregorian years. Show that 400 consecutive Gregorian years consists of an integral number of weeks. (*5 points*)
   (b) Prove that the probability that Christmas falls on a Wednesday is not equal to 1/7. (*5 points*)

Solution

(a) part 1

According to the given rules, any 400 consecutive Gregorian years will involve, 303 common years and 97 leap years. There are many ways to see this. One way is to count the number of leap years.

Rules (i-iv) imply that a year is a common year or a leap year.

Rule (ii) implies there are at least 96 leap years excluding years 100, 200, 300, and 400.

Rule (iv) indicates that 400 is a leap year.

So there are exactly 303 common years and 97 leap years in 400 consecutive Gregorian years.

(a) part 2

Thus any 400 consecutive Gregorian years will have a total of (400)(365)+97 days. This number divisible by 7, to see this just do the arithmetic, (400)(365)+97 = 146,097 and 146,097 divided by 7 is 20,871. Thus there are an integral number of weeks in any 400 consecutive Gregorian years.

(b)

Let N be the number of times Christmas falls on a Wednesday in a particular 400 consecutive Gregorian years. Then the probability that Christmas falls on Wednesday is N/400. Since 7 is not a factor of 400, $N/400 \neq 1/7$ for any integer N.

5. Each of the first 13 letters of the alphabet is written on the back of a card and the 13 cards are placed in a row in the order

$$A, B, C, D, E, F, G, H, I, J, K, L, M$$

The cards are then turned over so that the letters are face down.

The cards are rearranged and again placed in a row, but of course they may be in a different order. They are rearranged and placed in a row a second time and both rearrangements were performed exactly the same way. When the cards are turned over the letters are in the order

$$B, M, A, H, G, C, F, E, D, L, I, K, J$$

What was the order of the letters after the cards were rearranged the first time? (*10 points*)

Solution

The idea in the solution is to note that the second reordering doesn't fix any subset of the letters so the original reordering doesn't either.

We have a cycle where we know every other letter

$$A, -, C, -, F, -, G, -, E, -, H, -, D, -, I, -, K, -, L, -, M, -, B, -, A, -, \ldots$$

Any letter must be repeated 13 spaces later. So A appears after D, C appears after I, F appears after K, etc. Then the cycle looks like

$$A, I, C, K, F, L, G, J, E, M, H, B, D, A, \ldots$$

So

$$A, B, C, D, E, F, G, H, I, J, K, L, M$$

is initially rearranged to

$$D, H, I, B, J, K, L, M, A, G, C, F, E$$

then to

$$B, M, A, H, G, C, F, E, D, L, I, K, J$$