MMPC STUDENT CODE

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# FORTY-SIXTH ANNUAL
# MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

## Part II

December 4, 2002

## INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

| # | 1 | 2 | 3 | 4 | 5 | Total |
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1. (a) Show that for every positive integer $m > 1$, there are positive integers $x$ and $y$ such that $x^2 - y^2 = m^3$. (*5 points*)

   (b) Find all pairs of positive integers $(x, y)$ such that $x^6 = y^2 + 127$. (*5 points*)

☐ Check here if this solution is continued on page 7.

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2. (a) Let $P(x)$ be a polynomial with integer coefficients. Suppose that $P(0)$ is an odd integer and that $P(1)$ is also an odd integer. Show that if $c$ is an integer then $P(c)$ is not equal to 0. (*5 points*)

   (b) Let $P(x)$ be a polynomial with integer coefficients. Suppose that $P(1{,}000) = 1{,}000$ and $P(2{,}000) = 2{,}000$. Explain why $P(3{,}000)$ cannot be equal to 1,000. (*5 points*)

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3. Triangle $\triangle ABC$ is created from points $A(0,0)$, $B(1,0)$ and $C(1/2,2)$. Let $q$, $r$, and $s$ be numbers such that $0 < q < 1/2 < s < 1$, and $q < r < s$. Let $D$ be the point on $AC$ which has $x$-coordinate $q$, $E$ be the point on $AB$ which has $x$-coordinate $r$, and $F$ be the point on $BC$ that has $x$-coordinate $s$.

   (a) Find the area of triangle $\triangle DEF$ in terms of $q$, $r$, and $s$. (*6 points*)

   (b) If $r = 1/2$, prove that at least one of the triangles $\triangle ADE$, $\triangle CDF$, or $\triangle BEF$ has an area of at least $1/4$. (*4 points*)

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4. In the Gregorian calendar:

   (i) years not divisible by 4 are common years,

   (ii) years divisible by 4 but not by 100 are leap years,

   (iii) years divisible by 100 but not by 400 are common years,

   (iv) years divisible by 400 are leap years,

   (v) a leap year contains 366 days; a common year 365 days.

   From the information above:

   (a) Find the number of common years and leap years in 400 consecutive Gregorian years. Show that 400 consecutive Gregorian years consists of an integral number of weeks. (*5 points*)

   (b) Prove that the probability that Christmas falls on a Wednesday is not equal to 1/7. (*5 points*)

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5. Each of the first 13 letters of the alphabet is written on the back of a card and the 13 cards are placed in a row in the order

$$A, B, C, D, E, F, G, H, I, J, K, L, M$$

The cards are then turned over so that the letters are face down. The cards are rearranged and again placed in a row, but of course they may be in a different order. They are rearranged and placed in a row a second time and both rearrangements were performed exactly the same way. When the cards are turned over the letters are in the order

$$B, M, A, H, G, C, F, E, D, L, I, K, J$$

What was the order of the letters after the cards were rearranged the first time? (*10 points*)

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(continued solutions)

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.