FORTY-SIXTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 9, 2002

INSTRUCTIONS
(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Carefully read and follow the instructions printed on the answer sheet. Check to ensure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.
2. Do as many problems as you can in the 100 minutes allowed. When the proctor asks you to stop, please quit working immediately and turn in your answer sheet.
3. These problems may consider careful consideration. Do not be hasty in your judgments. You may work out ideas on scratch paper before selecting a response.
4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number of correct answers. You are advised to guess an answer in those cases where you cannot determine an answer.
5. For each of the questions, five different possible responses are provided. In some cases the fifth alternative is "(e) None of above." If you believe none of the first four alternatives is correct, mark (e) in such cases.
6. Any scientific or graphing calculator is permitted on Part I. Unacceptable machines include computers, PDAs, pocket organizers and similar devices. All problems will be solvable with no more technology than a scientific calculator. The Exam Committee makes every effort to structure the test to minimize the advantage of a more powerful calculator.
7. No one is permitted to explain to you the meaning of any question. Do not ask anyone to violate the rules of the competition. If you have questions concerning the instructions, ask them now.
8. You may now open the test booklet and begin.

1. Assume that $w \neq 0$. Simplify the expression $\left(-\frac{w}{2}\right)^{-3}\left(\frac{8}{w}\right)^{-1/3}$.

(a) $-\dfrac{w^{10/3}}{16}$ (b) $16w^{8/3}$ (c) $-\dfrac{4}{w^{8/3}}$ (d) -4 (e) $\dfrac{4}{w^{8/3}}$

2. One side of a rectangle is 5 meters long. Express the area A of the rectangle as a function of its perimeter p.

(a) $A = 5p$ (b) $A = \dfrac{p^2 - 100}{16}$ (c) $A = \dfrac{5}{2}p + 25$ (d) $A = 5p - 25$ (e) $A = \dfrac{5}{2}p - 25$

3. Suppose $ax + 3y = 9$ and $5x + by = 6$ are both equations of the same straight line. What is a?

(a) $\dfrac{2}{15}$ (b) $\dfrac{15}{2}$ (c) $\dfrac{7}{15}$ (d) $\dfrac{15}{7}$ (e) $\dfrac{2}{9}$

4. Suppose that three lines have the same y-intercept and that their slopes are $1/3$, $1/5$, $1/7$. If the sum of the x-intercepts of these lines is 1000 find their common y-intercept.

(a) $\dfrac{200}{3}$ (b) $\dfrac{3}{200}$ (c) $-\dfrac{3}{200}$ (d) $-\dfrac{200}{3}$ (e) 200

5. A point Q is said to divide the line segment Q_1Q_2 in the ratio $r_1 \colon r_2$ if Q lies between Q_1 and Q_2 and $|QQ_1| \,/\, |QQ_2| = r_1/r_2$. Determine the coordinates of the point that divides the line segment connecting $(2, 3)$ to $(5, -4)$ in the ratio 7:3.

(a) $(3.5, -0.5)$ (b) $(2.9, 1.1)$ (c) $(4.1, -1.9)$ (d) $(0.7, 0.0)$ (e) $(9.0, -13.3)$

6. Two circles are externally tangent. Their radii are 2 and 4. A line is drawn that is tangent to the circles at points P and Q and $P \neq Q$. What is the distance $|PQ|$?

(a) $4\sqrt{2}$ (b) $2\sqrt{10}$ (c) $2\sqrt{3}$ (d) 8 (e) None of the above.

7. Simplify the expression $\dfrac{1}{2 - \frac{1}{3+\sqrt{5}} + \frac{1}{3-\sqrt{5}}}$.

(a) $\dfrac{1}{5}$ (b) $-4 + 2\sqrt{5}$ (c) $\dfrac{1}{1+\sqrt{5}}$ (d) $\dfrac{1}{1+2\sqrt{5}}$ (e) None of the above.

8. Suppose that $x - y = 0.0002$. Which of the following conditions implies that $x^3 - y^3 > 1000$?

   (a) $x > 10$ and $y < -10$
   (b) $x^2 + y^2 > 3{,}000{,}000$
   (c) $x > 1{,}000$ and $y > 1{,}000$
   (d) $x < -1300$ and $y < -1300$
   (e) None of the above.

9. Find the number of pairs (a, b) of positive integers such that $3a + 2b = 100$.

   (a) 15 (b) 16 (c) 17 (d) 18 (e) 19

10. An athlete runs the first 400 meters of an 800 meter race at an average speed of 6 m/s. How fast must the athlete run the second half of the race in order to achieve an average speed of 7 m/s for the entire race?

   (a) $\frac{800}{\frac{800}{7} - \frac{400}{6}}$ m/s (b) $\frac{400}{\frac{800}{6} - \frac{400}{7}}$ m/s (c) $\frac{400}{\frac{800}{7} - \frac{400}{6}}$ m/s (d) $\frac{800}{6}$ m/s (e) $\frac{400}{7}$ m/s

11. Suppose a and b are non-negative integers and that $3^{a+6}\,6^{b+2}$ is a perfect cube. Find the smallest value of ab.

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

12. If $(a + b + c + d + e + f + g + h + i + j)^2$ is expanded and simplified, what is the sum of the coefficients?

   (a) 5^{10} (b) 10^5 (c) 10^{10} (d) 10^2 (e) 2^{10}

13. Determine the eighteenth number in this sequence: 4, 9, 25, 49, 121, 169, 289, 361, ...

   (a) 871 (b) 2809 (c) 3481 (d) 3721 (e) 4489

14. How many positive integers less than 122 are not divisible by either 3 or 8?

   (a) 50 (b) 55 (c) 70 (d) 71 (e) 187

15. A triangle has vertices A, B, and C. The respective opposite sides have length a, b, and c. The expression $b\cos C + c\cos B$ is equal to:

(a) a (b) $a\cos A$ (c) $\sin^2 A + \cos^2 A$ (d) $bc\cos(B+C)$ (e) $a\sin C$

16. Each of three circles is externally tangent to the other two. The radii of these circles are 2 cm, 4 cm, and 11 cm. A triangle is created using the centers of these circles as vertices. Which of the following is closest to the measure of the largest angle of the triangle (given in degrees and minutes)?

(a) $59°16'$ (b) $82°38'$ (c) $97°22'$ (d) $97°37'$ (e) No triangle exists that fits these conditions.

17. If $x + y > 9$ and $2x - y < 3$, then

(a) $x > 4$ and $y > 5$
(b) $x > 4$ and $y < 5$
(c) $x < 4$ and $y > 5$
(d) $x < 4$ and $y < 5$
(e) None of the above.

18. Points A and B lie on a circle of radius 2 and center at O and the measure of $\angle AOB$ is $90°$. What is the area of the smaller region bounded by the circle and the chord joining A to B?

(a) π (b) 2 (c) $\frac{\pi}{8}$ (d) 4π (e) $\pi - 2$

19. Solve the inequality $\dfrac{1}{x+2} > \dfrac{2}{x+3}$.

(a) $x > -1$
(b) $-2 < x < -1$
(c) $x < -3$ or $-2 < x < -1$
(d) $-2 < x < -1$ or $x > -1$
(e) $x < -3$ or $x > -1$

20. Evaluate $\dfrac{2+4+6+\cdots+4{,}002+4{,}004}{1+3+5+\cdots+4{,}001+4{,}003}$.

(a) $\frac{2{,}003}{2{,}002}$ (b) $\frac{1{,}002}{1{,}001}$ (c) $\frac{2{,}005}{2{,}002}$ (d) $\frac{1{,}003}{1{,}001}$ (e) $\frac{2{,}007}{2{,}002}$

21. Three regular hexagons are constructed, each with one side coinciding with one side of a right triangle. Let C denote the area of the hexagon constructed on the hypotenuse and let A and B denote the areas of the hexagons constructed on the other sides. Which of the following statements is true?

 (a) $C = A + B$
 (b) $C > A + B$
 (c) $C^2 = A^2 + B^2$
 (d) $C^2 > A^2 + B^2$
 (e) None of the above.

22. The time it takes for a pendulum to complete one swing is called the period of the pendulum. The period is known to be proportional to the square root of the length of the pendulum. A pendulum that is 6 feet long has a period of 2.7 seconds. Which of the following is closest to the length of a pendulum that has a period of 2 seconds?

 (a) 1.8 feet (b) 3.3 feet (c) 4.4 feet (d) 7.2 feet (e) 10.9 feet

23. Solve the inequality $|x + 1| + |x - 1| > 3$.

 (a) $|x| > 4$ (b) $|x| > \sqrt{2}$ (c) $x > \frac{3}{2}$ (d) $|x| > \frac{3}{2}$ (e) None of the above.

24. Find the sum of all integers obtained by permuting the digits in 12,345.

 (a) 6,999,930 (b) 3,999,960 (c) 3,699,990 (d) 9,999,630 (e) 3,999,906

25. The makers of a weight-loss medicine advertise that the use of their product will result in a loss of at least 5% of the purchaser's weight every 6 months. A person who weighs 250 pounds purchases and intends to use the medicine for a year. At least how many pounds should this person expect to lose as a result of the medicine?

 (a) 23.791 (b) 23.885 (c) 24.375 (d) 25 (e) 225.625

26. For what values of a and b is $p(x) = x^2 + 1$ a factor of $q(x) = x^{10} + ax + b$?

 (a) $a = 0$, $b = -1$
 (b) $a = i$, $b = 2$
 (c) $a = 0$, $b = 0$
 (d) $a = x^7$, $b = 1$
 (e) None of the above.

27. Find the sum of the digits of the product of 999,999,999 and 123,456,789,123,456,789,123,456,789.

(a) 252 (b) 242 (c) 234 (d) 233 (e) 243

28. The digits 4, 4, 5, 7 are randomly arranged to form a 4-digit number. What is the probability that the sum of the first and last digits is even?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{4}$ (e) $\frac{3}{4}$

29. A triangle has vertices at A, B, and C. The respective opposite sides have lengths a, b, and c. If the angle at B is twice the angle at C, then $b^2 - c^2$ is always equal to:

(a) a^2 (b) $-a^2$ (c) ac (d) ab (e) $3c^2$

30. In a sequence, every term after the second term is twice the sum of the two preceding terms. If the fifth term is 20 and the seventh term is 50, find the ninth term.

(a) 120 (b) 130 (c) 210 (d) 230 (e) 320

31. In triangle $\triangle ABC$, $AB \perp BC$. The point D lies between A and B, E lies between B and C, and F lies between A and C. The line segment $DF \perp AB$ and $EF \perp BC$. If $|AB| = 10$, $|BC| = 7$, and $|AD| = x$ then the area of the rectangle $DBEF$ is:

(a) $\frac{7x(10-x)}{10}$ (b) $\frac{(10-x)(7+x)}{10}$ (c) $\frac{10x(7-x)}{7}$ (d) $\frac{70-x^2}{7}$ (e) None of the above.

32. There are four integers a, b, c, d such that $d = 5a + 3b + 5c$ and $d = 4a + 5b + 4c$. Suppose $131 \leq d \leq 150$. What is the value of $a + b + c$?

(a) 11 (b) 22 (c) 33 (d) 44 (e) 55

33. A circle is inscribed in a right triangle whose sides are 5, 12, and 13. What is the area of the circle?

(a) $\frac{12\pi}{5}$ (b) 4π (c) $2\pi\sqrt{5}$ (d) $3\pi\sqrt{2}$ (e) None of the above.

34. If $a > 0$, what are the rectangular coordinates of the center of the ellipse $3x^2 + 5y^2 + 6x - 20y = a$?

(a) $(1, -2)$ (b) $(a-1, a+2)$ (c) $(-1, 2)$ (d) $(2a, a)$ (e) $(a+2, 1-a)$

35. Two chords of a circle intersect inside the circle. One of the chords is divided by the second chord into segments that are 2 cm and 7 cm long. If the shorter of the two segments of the second chord is 3 cm long what is the length of the other segment?

(a) $\frac{7}{2}$ cm (b) $\frac{9}{2}$ cm (c) $\frac{11}{3}$ cm (d) $\frac{13}{3}$ cm (e) None of the above.

36. A sailboat leaves Pentwater at 6 o'clock in the evening and sails due west into Lake Michigan for 2 hours. The boat then turns and sails on a course that is 33° west of due north for 2 hours. If the boat has sailed at a constant speed of 10 miles per hour, which of the following is the best approximation to its distance from Pentwater at 9 o'clock?

(a) 18 miles (b) 21 miles (c) 24 miles (d) 27 miles (e) 35 miles

37. What is the units digit in the decimal expansion of $(2002)^{2002}$?

(a) 2 (b) 4 (c) 6 (d) 8 (e) None of the above.

38. Each of three coins has been painted on its two sides. The first coin is red on one side and blue on the other. The second coin is blue on one side and green on the other. The third coin is green on one side and red on the other. If the three coins are tossed in the air, what is the probability that all three colors will appear face up when they land?

(a) $\frac{1}{4}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) $\frac{5}{8}$ (e) $\frac{3}{4}$

39. Find the sum of all the values of x that satisfy the following equation.

$$(x^2 + 999x + 1000)^{x^2+1000x+999} = 1$$

(a) -1999 (b) 1999 (c) 999 (d) -1000 (e) 1000

40. Consider the points inside the circle $x^2 + y^2 = 16$ that are the midpoints of vertical line segments drawn between the circle and the x-axis. These points lie on:

(a) a circle (b) a parabola (c) a line (d) an ellipse (e) None of the above.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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