Forty-Fifth Annual

Michigan Mathematics Prize Competition

Sponsored by the

Michigan Section of the Mathematical Association of America

Part II solutions

December 5, 2001

1. At m minutes after 12:00, the hour hand will be pointing to $\frac{m}{12}$ minutes past 12:00. At $\frac{m}{12}$ minutes past 1:00, the hour hand will be pointing to $5 + \frac{m}{144}$ minutes past 12:00. Thus, one symmetric natural placement of the hands will occur when $m = 5 + \frac{m}{144}$. The solution $m = \frac{720}{143} \approx 5.035$ gives the number of minutes past 12:00 when a symmetric natural placement of the hands will occur.

2. Suppose the repetition of the digits 2001 starts to the right of the kth digit to the right of the decimal point. Then $\frac{10^k m}{n} = q \pm 0.200120012001\ldots$ for some integer q. Notice that $0.200120012001\ldots = \frac{2001}{9999} = \frac{667}{3333} = \frac{667}{33 \cdot 101}$. Thus $\frac{10^k m \pm nq}{n} = \frac{667}{33 \cdot 101}$. This gives $667n = 33 \cdot 101(10^k m \pm nq)$. Since the prime 101 does not divide 667, it must divide n.

3. a. Assume that you have a scheme for asking seven yes-or-no questions from which you can identify the secret number $i \in \{0, 1, \ldots, 15\}$ even if there is a lie in the answers. For each $i \in \{0, 1, \ldots, 15\}$, let $\sigma_i = (a_i, b_i, c_i, d_i, e_i, f_i, g_i)$ be the sequence of truthful answers to your seven questions if the secret number is i. Use 1 for the answer "yes" and 0 for the answer "no".

   We claim that $\sigma_0, \sigma_1, \ldots, \sigma_{15}$ provide a solution to Question 2. Suppose to the contrary that σ_i and σ_j differ in fewer than three coordinates for some $i \neq j$. Then you can obtain the same sequence σ by changing at most one coordinate of σ_i and at most one coordinate of σ_j. Therefore with at most one lie in responding to the questions, you can get the sequence σ for both the secret numbers i and j. This contradicts the ability of your seven questions to differentiate between i and j if a lie is allowed.

b. Question 1: Start by writing the secret number i in binary as $i = p+2q+4r+8s$ where each of p, q, r, s is either 0 or 1. Then the seven questions are:

1. Is p equal to 1?
2. Is q equal to 1?
3. Is r equal to 1?
4. Is s equal to 1?
5. Is $p+q+r$ odd?
6. Is $p+q+s$ odd?
7. Is $q+r+s$ odd?

Question 2: Let the 16 points be those that represent truthful answers to these seven questions for the secret number being $0, 1, \ldots, 15$. Specifically,

$$\begin{aligned}
\sigma_0 &= (0,0,0,0,0,0,0)\\
\sigma_1 &= (1,0,0,0,1,1,0)\\
\sigma_2 &= (0,1,0,0,1,1,1)\\
\sigma_3 &= (1,1,0,0,0,0,1)\\
\sigma_4 &= (0,0,1,0,1,0,1)\\
\sigma_5 &= (1,0,1,0,0,1,1)\\
\sigma_6 &= (0,1,1,0,0,1,0)\\
\sigma_7 &= (1,1,1,0,1,0,0)\\
\sigma_8 &= (0,0,0,1,0,1,1)\\
\sigma_9 &= (1,0,0,1,1,0,1)\\
\sigma_{10} &= (0,1,0,1,1,0,0)\\
\sigma_{11} &= (1,1,0,1,0,1,0)\\
\sigma_{12} &= (0,0,1,1,1,1,0)\\
\sigma_{13} &= (1,0,1,1,0,0,0)\\
\sigma_{14} &= (0,1,1,1,0,0,1)\\
\sigma_{15} &= (1,1,1,1,1,1,1)
\end{aligned}$$

4. a.

$$\begin{aligned}
\sin 3x &= \sin(2x+x)\\
&= \sin 2x\cos x + \cos 2x \sin x\\
&= (2\sin x\cos x)(\cos x) + \cos 2x\sin x\\
&= (\sin x)(2\cos^2 x + \cos 2x)\\
&= (\sin x)(1+\cos^2 x - \sin^2 x + \cos 2x)\\
&= (\sin x)(1+\cos 2x + \cos 2x)\\
&= (\sin x)(1+2\cos 2x).
\end{aligned}$$

Or:

$$\begin{aligned}
\sin 3x &= \sin(2x+x)\\
&= \sin 2x\cos x + \cos 2x\sin x\\
&= \sin 2x\cos x - \cos 2x\sin x + 2\cos 2x\sin x\\
&= \sin(2x-x) + 2\cos 2x\sin x\\
&= (\sin x)(1+2\cos 2x).
\end{aligned}$$

b. For some integer $n > 1$, assume that

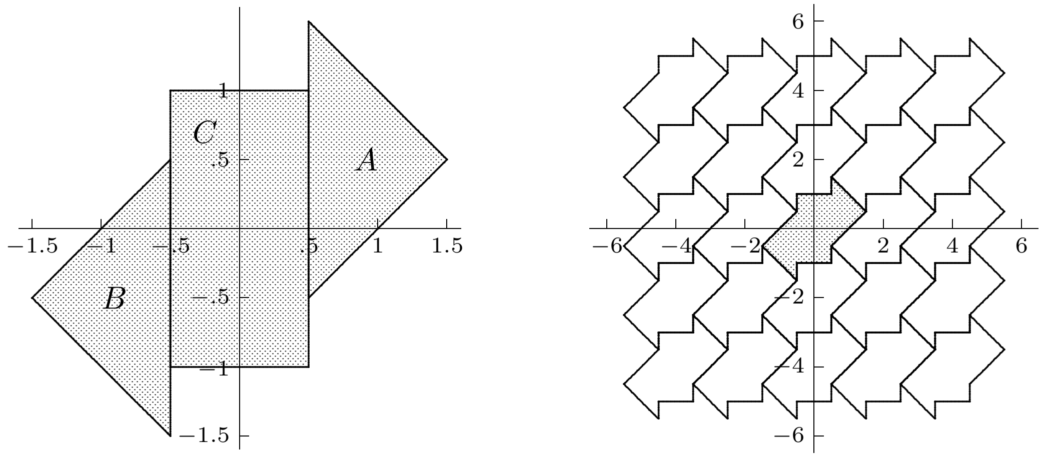
$$(\sin x)\bigl(1+2\cos 2x+2\cos 4x+\cdots+2\cos(2(n-1)x)\bigr) = \sin\bigl((2n-1)x\bigr).$$

Then for this value of n, we have

$$\begin{aligned}
&(\sin x)(1+2\cos 2x+2\cos 4x+\cdots+2\cos 2nx)\\
&\qquad= (\sin x)\bigl(1+2\cos 2x+2\cos 4x+\cdots+2\cos(2(n-1)x)\bigr) + (\sin x)(2\cos 2nx)\\
&\qquad= \sin\bigl((2n-1)x\bigr) + (\sin x)\bigl(2\cos 2nx\bigr)\\
&\qquad= (\sin 2nx)(\cos x) - (\cos 2nx)(\sin x) + 2(\cos 2nx)(\sin x)\\
&\qquad= (\sin 2nx)(\cos x) + (\cos 2nx)(\sin x)\\
&\qquad= \sin(2nx+x)\\
&\qquad= \sin\bigl((2n+1)x\bigr).
\end{aligned}$$

Therefore, by the Principle of Mathematical Induction, the identity holds for all positive integers n.

5. a. On the left is a picture of Ω. On the right at a different scale is a picture of how Ω can be translated to tile the plane.



b. All the points (x, y) of the region bounded by triangle A satisfies $y > -2x$, so $G = -1$ and (x, y) is mapped to $(x - 1, x + y - 1)$. In particular, the three vertices $(0.5, 1.5)$, $(1.5, 0.5)$, and $(0.5, -0.5)$ of triangle A are mapped to $(-0.5, 1.0)$, $(0.5, 1.0)$, and $(-0.5, -1.0)$, respectively. These are the vertices along the bottom and right sides of rectangle C. Thus, under this transformation, the region bounded by triangle A is mapped to the triangular region W in the diagram below. Similarly, for triangle B we have $G = 1$ and the region bounded by triangle B is mapped to the triangular region X in the diagram.

Break up the rectangular region into a trianglar region Y below the line $y = -2x$ and a triangular region Z above this line. The transformation (with $G = 1$) maps region Y to the region bounded by triangle A and (with $G = -1$) maps region Z to the region bounded by triangle B. Likewise, the line segment between $(-0.5, 1.0)$ and $(0.5, -1.0)$ is mapped (with $G = 0$) to the line segment between $(-0.5, 0.5)$ and $(0.5, -0.5)$, which is in Ω.

Notice that $(-1, 0)$ is transformed to $(-1 + 1, -1 + 0 + 1) = (0, 0)$, and $(1, 0)$ is transformed to $(1 - 1, 1 + 0 - 1) = (0, 0)$. Of course, $(0, 0)$ is also transformed to $(0, 0)$.