MMPC CODE NUMBER

FORTY-FIFTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

December 5, 2001

INSTRUCTIONS
(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes (1 hour and 40 minutes) for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page clearly write the **problem number**. If you use additional paper for your answer, check the appropriate box and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

6. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference material, computation aids, or any other aid is likewise prohibited. Please note that **calculators are not allowed** on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. Please do not return scratch paper containing routine numerical calculations.

7. You may now open the test booklet.

#	**Score**
1	
2	
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4	
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$\sum$	

1. A clock has a long hand for minutes and a short hand for hours. A placement of those hands is *natural* if you will see it in a correctly functioning clock. So, having both hands pointing straight up toward 12 is natural and so is having the long hand pointing toward 6 and the short hand half-way between 2 and 3. A natural placement of the hands is *symmetric* if you get another natural placement by interchanging the long and short hands. One kind of symmetric natural placement is when the hands are pointed in exactly the same direction.

   Are there symmetric natural placements of the hands in which the two hands are not pointed in exactly the same direction? If so, describe one such placement. If not, explain why none are possible.

Check here if your solution is continued on the inside back cover (page 7) _____
Check here if your solution is continued on the inserted additional paper _____

2. Let $\frac{m}{n}$ be a fraction such that when you write out the decimal expansion of $\frac{m}{n}$, it eventually ends up with the four digits 2001 repeated over and over and over. Prove that 101 divides n.

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3. Consider the following two questions:

Question 1: I am thinking of a number between 0 and 15. You get to ask me seven yes-or-no questions, and I am allowed to lie at most once in answering your questions. What seven questions can you ask that will always allow you to determine the number? Note: You need to come up with seven questions that are independent of the answers that are received. In other words, you are not allowed to say, "If the answer to question 1 is yes, then question 2 is XXX; but if the answer to question 1 is no, then question 2 is YYY."

Question 2: Consider the set S of all seven-tuples of zeros and ones. What sixteen elements of S can you choose so that every pair of your chosen seven-tuples differ in at least three coordinates?

a. These two questions are closely related. Show that an answer to Question 1 gives an answer to Question 2.

b. Answer either Question 1 or Question 2.

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4. You may wish to use the angle addition formulas for the sine and cosine functions:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

a. Prove the identity $(\sin x)(1 + 2\cos 2x) = \sin(3x)$.

b. For any positive integer n, prove the identity

$$(\sin x)(1 + 2\cos 2x + 2\cos 4x + \cdots + 2\cos 2nx) = \sin\big((2n+1)x\big).$$

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5. Define the set Ω in the xy-plane as the union of the regions bounded by the three geometric figures: triangle A with vertices $(0.5, 1.5)$, $(1.5, 0.5)$ and $(0.5, -0.5)$; triangle B with vertices $(-0.5, -1.5)$, $(-1.5, -0.5)$ and $(-0.5, 0.5)$; and rectangle C with corners $(0.5, 1.0)$, $(-0.5, 1.0)$, $(-0.5, -1.0)$, and $(0.5, -1.0)$.

   a. Explain how copies of Ω can be used to cover the xy-plane. The copies are obtained by translating Ω in the xy-plane, and copies can intersect only along their edges.

   b. We can define a transformation of the plane as follows: map any point (x, y) to $(x + G, x + y + G)$, where $G = 1$ if $y < -2x$, $G = -1$ if $y > -2x$, and $G = 0$ if $y = -2x$. Prove that every point in Ω is transformed into another point in Ω, and that there are at least two points in Ω that are transformed into the same point.

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