Forty-Fifth Annual

Michigan Mathematics Prize Competition
Solutions to Part I

October 10, 2001

1. b) An equation of the parallel line is $2x + 7y = c$. Since $(3, -1)$ is on the line, $c = 2 \cdot 3 + 7(-1) = -1$. Thus the equation is $2x + 7y = -1$. When $x = 0$, we get the y-intercept $y = -\frac{1}{7}$.

2. d) There are $24 - 10 = 14$ females. Of these, $14 - 11 = 3$ are left-handed. Thus there are $4 + 3 = 7$ left-handed students in the class.

3. b) We have $3^2 + 2b + c = -2$ and $5^2 + 5b + c = 10$. Subtracting the first equation from the second gives $16 + 2b = 12$, or $b = -2$. Thus $c = -2 - 9 - 3(-2) = -5$. Now $P(2) = 2^2 - 4 \cdot 2 - 5 = -5$.

4. b) Let $r = 4000 \cdot 5280 = 21120000$ denote the radius of the earth in feet. Consider the right triangle with vertices at the center of the earth, the woman's head, and the point she sees on the horizon. The distance to the horizon is $\sqrt{(r+5)^2 - r^2} = \sqrt{10r + 25} \approx 14533$ or $\frac{14533}{5280} \approx 2.75$ miles.

5. d) There are four equally likely groups of three. Two of these contain both of the cat lovers.

6. e) Shift the graph of $y = \frac{1}{x}$ one unit to the right to get the graph of $y = \frac{1}{x-1}$. The shifted graph is above the original graph if $x < 0$ or $1 < x$.

7. c) Consider the equilateral triangle with vertices at the centers of the three circles. The altitude of this triangle is $\sqrt{3}$. Thus the distance from the center of this triangle to a vertex is $\frac{2}{3}\sqrt{3} = \frac{2}{\sqrt{3}}$. So the radius of the circumscribing circle is $1 + \frac{2}{\sqrt{3}}$.

8. a) Multiply both sides of the equation by $x(x+a)(x-a)$ to get $x^2 - a^2 = x(x-a) + x(x+a)$. Thus $x^2 - a^2 = 2x^2$, or $x^2 + a^2 = 0$. Since $a^2 > 0$, there are no real solutions.

9. c) The player now has (.300)200=60 hits, and needs to finish the season with (.320)500=160 hits. So 100 additional hits are needed.

10. e) For example, if Gonzales had fewer at-bats than Fielder in 1987, his batting average that year would be weighted less heavily in the average for the combined period while Fielder's would be weighted more heavily. This could result in this unexpected result.

11. d) Let x be the number of pages. Clearly $100 \leq x \leq 999$. So the number of digits is $9 + 2 \cdot 90 + 3(x - 99) = 651$. Thus $x = 253$.

12. b) Of the 36 possible outcomes in one toss of two dice, 6 are doubles. Thus the probability of doubles in one toss is $\frac{6}{36} = \frac{1}{6}$. The probability of three doubles in three rolls is $\left(\frac{1}{6}\right)^3 = \frac{1}{216}$.

13. a) Since $1, x, y$ form an arithmetic progression, $y = x + (x - 1) = 2x - 1$. Thus $1 + x^2 = (2x-1)^= 4x^2 - 4x + 1$. That is, $3x^2 = 4x$. So either $x = 0$ or $x = \frac{4}{3}$. The possible ordered pairs are $(0, -1)$ and $\left(\frac{4}{3}, \frac{5}{3}\right)$.

14. b) The arch has an equation of the form $y = 55 - ax^2$. The feet on the x-axis at ± 100 gives $0 = 55 - 100^2 x$, or $a = .0055$. Thus the equation is $y = 55 - .0055x^2$. At $x = 40$ we have $y = 55 - .0055(40^2) = 46.2$.

15. c) Among the 64 ordered pairs of outcomes of the two dice, there are 8 that yield a sum of 8. For any other sum, there are fewer than 8 ordered pairs.

16. d) We can pair up the divisors of n less than $\sqrt{n}$ with the divisors of n greater than $\sqrt{n}$ by matching a divisor $d < \sqrt{n}$ with $\frac{n}{d} > \sqrt{n}$. This accounts for an even number of divisors. There is one more divisor, $\sqrt{n}$ itself, if and only if n is a perfect square.

17. c) The center of one pipe and the center of one touching it in the row above it are 10 apart (all units are in centimeters). These centers are displaced horizontally by 5. Thus they are displaced vertically by $\sqrt{10^2 - 5^2} = \sqrt{75}$. Thus the height of the pile is $5 + 4\sqrt{75} + 5 \approx 44.64$.

18. d) Square the first equation and subtract the second from it to find that $2xy = (x^2 + 2xy + y^2) - (x^2 + y^2) = 1 - 2 = -1$, so $xy = -\frac{1}{2}$. Now $x^3 + y^3 = (x + y)(x^2 - xy + y^2) = 1(2 - xy) = 2 - \left(-\frac{1}{2}\right) = \frac{5}{2}$.

19. d) Since x is negative, $x+|x|=0$. Thus $y+|y|=0$. Therefore $y\le 0$. The solution set is the third quadrant of the plane (excluding the y-axis but including the negative x-axis).

20. e) Let r denote the radius of the circle. We want $\pi r^2+2\pi r-m=0$. Thus, $r=\frac{-2\pi\pm\sqrt{4\pi^2+4\pi m}}{2\pi}=\frac{-2\pi\pm 2\pi\sqrt{1+\frac{m}{\pi}}}{2\pi}=-1\pm\sqrt{1+\frac{m}{\pi}}$. Even if $m>-\pi$, we have only one positive value of r. That's what we get for adding length and area!

21. d) The first 24 start with Q. There are followed by 24 starting with R, 24 starting with S, and 24 starting with T. The next one is the 97th. It is $UQRST$, the first word starting with U.

22. e) Use induction: $f^1(2)=(-1)^1 2^{3^1}$; and if $f^k(2)=(-1)^k 2^{3^k}$, then $f^{k+1}(2)=f(f^k(2))=f((-1)^k 2^{3^k})=-\left((-1)^k 2^{3^k}\right)^3=(-1)^{3k+1}2^{3^k\cdot 3}=(-1)^{2k}(-1)^{k+1}2^{3^{k+1}}$. Thus, $f^{1000}(2)=(-1)^{1000}2^{3^{1000}}=2^{3^{1000}}$.

23. b) The primes between 100 and 120 are 101, 103, 107, 109, and 113. Notice that $119=7\cdot 17$.

24. e) The distance between neighboring chairs equals the distance from a chair to the center of the Ferris wheel.

25. c) Among the $12\cdot 11\cdot 10$ equally likely ordered triples of three balls, there are $5\cdot 4\cdot 3$ ways to get one ball of each color in any of the $3!=6$ orderings of the colors. Thus the probability is $\frac{5\cdot 4\cdot 3\cdot 6}{12\cdot 11\cdot 10}=\frac{3}{11}$.

26. a) Let h be the height of the cone in one hour. Then the radius of the base will be $2h$ and the volume of the cone will be $\frac{1}{3}\pi(2h)^2h$. This is equal to $\frac{1}{3}\pi 6^2\cdot 3+60\pi=96\pi$. Solving this equation for h yields $h=\sqrt[3]{72}$.

27. b) With one blue ball on the left, there are 5 positions for the other blue ball. Every time we move the leftmost blue ball one position to the right, the number of positions for the other blue ball decreased by one. Thus we have $5+4+3+2+1=15$ arrangements.

28. b) Notice that $x^4-2ax^2+a^2-a=(x^2-a)^2-a$. Thus if this is equal to 0, then $x=\pm\sqrt{a\pm\sqrt{a}}$. Let $a=1$ and $x=\sqrt{2}$ to rule out a), c), and d). Let $a=9$ and $x=\sqrt{12}$ to rule out e).

29. b) The square whose diagonal is 1 fits properly inside the other square and the circle. Its area is $\left(\frac{1}{\sqrt{2}}\right)^2=\frac{1}{2}$, whereas the area of the hexagon (broken into 6 equilateral triangles of side length $\frac{1}{2}$) is $6\cdot\frac{1}{2}\cdot\frac{1}{2}\frac{\sqrt{3}}{4}=\frac{3\sqrt{3}}{8}>\frac{1}{2}$.

30. a) Divide both expressions by a^bb^a. The left side becomes $a^{a-b}b^{b-a}=\left(\frac{a}{b}\right)^{a-b}$. This is greater than 1 since $a>b$.

31. d) B can contain exactly one odd number in 3 ways or it can contain all three odd numbers. For each of these four possibilities, either A or B can contain 2, and either A or B can contain 4.

32. a) There is one change in sign in the coefficients of $p(x)=x^6+4x^3+x-2$, and one change in sign of the coefficients of $p(-x)=x^6-4x^3-x-2$. Thus there are at most 2 real roots (at most one positive and at most one negative).

33. e) There are 6 choices for the first 0, then 5 for the second, 4 for the third, 3 for the fourth, 2 for the fifth, and 1 for the last.

34. d) If x has a factor in common with n, then so will all multiples of x. Thus $x=17$ is the only candidate, and in fact $17(-196)=-3332=-1\cdot 3333+1$.

35. e) The square of an even number is divisible by 4; the square of an odd number leaves a remainder of 1 when divided by 4.

36. d) One of n, $n-1$, and $n-2$ is divisible by 3. So we cannot have n being even. We also cannot have $n-1$ being divisible by 4.

37. b) Choose one of the 2^{n-1} subsets of $\{1,2,\ldots,n-1\}$. Put these in increasing order to the left of n, and put the rest of the numbers to the right of n in decreasing order.

38. c) The number of intersections is $k(n-k)$. If we temporarily treat k as a real variable, we can plot this as a parabola with a maximum at $k=\frac{n}{2}$.

39. b) Forget about the other 49 cards. There are $3!=6$ equally likely ways to order the two, three, and four of clubs. Only one of these will have these cards in increasing order.

40. d) There was a total of $30\cdot 80=2400$ points. The teacher threw out $100+32=132$ points. The remaining 28 exams have an average score of $\frac{2400-132}{28}=\frac{2268}{28}=81$.