Forty-Fourth Annual

Michigan Mathematics Prize Competition

Solutions to Part I

October 11, 2000

1. a) In May the index went from I to $.8I$. In June it went from $.8I$ to $(1.15)(.8I) = .92I$.

2. b) The sum of all 60 numbers is $60 \cdot 24 = 1440$. The sum of the 58 numbers is $58 \cdot 25 = 1450$. Thus the sum of the two numbers is -10.

3. b) Dan paid $\frac{9.95}{\pi \cdot 38^3} \approx .0021933$. Bill paid $\frac{8.95}{2\pi \cdot 26^2} \approx .0021071$. Phil paid $\frac{7.95}{3\pi \cdot 20^2} \approx .0021088$. Ed paid $\frac{6.95}{4\pi \cdot 16^2} \approx .0021604$.

4. a) Among the integers from 0 to 80, the 9 possible remainders each occur 9 times. Thus all 9 remainders are equally likely when the sum of two numbers is divided by 9.

5. a) $f(2) = \frac{1}{3}$; $f\left(\frac{1}{3}\right) = \frac{-2/3}{4/3} = -\frac{1}{2}$; $f\left(-\frac{1}{2}\right) = \frac{-3/2}{1/2} = -3$.

6. b) We want to find t such that $1.5t + 15 = .8t + 18$. This equation gives $t = \frac{3}{.7} \approx 4.2857$. To the nearest minute, this is 4 hours and 17 minutes after 6 am, or 10:17 am.

7. e) The prime factorizations of these numbers are $957 = 3 \cdot 11 \cdot 29$ and $958 = 2 \cdot 479$. So $q = 1 + 3 + 11 + 29 + 33 + 87 + 319 + 957 = 1440$ and $r = 1 + 2 + 479 + 958 = 1440$.

8. d) There are $15 \cdot 14 \cdot 13 = 2730$ ordered triples of distinct numbers from this set. Of these, there are $10 \cdot 9 \cdot 8 = 720$ ordered triples of even numbers. Thus the probability is $\frac{720}{2730} = \frac{24}{91}$.

9. a) Let x be the side length of the building. An equation for the volume of soil removed is $(x+4)^2 - x^2 = 200$. Thus $8x + 16 = 200$, or $x = 23$.

10. e) Since $6^2 + 8^2 = 10^2$, the angle opposite the longest side is a right angle. Thus the longest side is a diameter of the circle. So the radius is 5.

11. a) The equation is saying that the distance from (x, y) to $(4, 0)$ plus the distance from (x, y) to $(0, -3)$ is the constant 7. Since the distance between $(4, 0)$ and $(0, -3)$ is 5, which is less than 7, the graph is an ellipse.

12. c) The slope of L_1 is $\frac{10}{12} = \frac{5}{6}$. Thus the slope of L_2 is $-\frac{6}{5}$. The point-slope equation for L_2 is $y - \frac{11}{6} = -\frac{6}{5}(x - \frac{1}{2})$. When $y = \frac{1}{30}$, we have $\frac{1}{30} - \frac{11}{6} = -\frac{6}{5}\left(x - \frac{1}{2}\right)$. So $x - \frac{1}{2} = -\frac{5}{6}\left(-\frac{54}{30}\right) = \frac{3}{2}$, or $x = 2$.

13. c) $x^2 - y^2 = 0$ represents a pair of lines with slopes ± 1, intersecting at the origin; $(x-a)^2 + y^2 = 1$ is a circle centered at $(a, 0)$. For large values of a, the circle is disjoint from the line. As a decreases to 0, the circle becomes tangent to the lines at 2 points, then crosses both lines at 4 points, then crosses both lines at the origin (3 solutions), and finally has a solution in each of the 4 quadrants. A symmetric pattern occurs for negative values of a.

14. b) If we cut apart all the triangles, we would have $\cdot 20 = 60$ vertices. In the assembles polyhedron, each vertex is shared with 5 triangles. Hence the polyhedron has $\frac{60}{5} = 12$ vertices

15. d) Let d be the distance north. Then $\cos 40^\circ = \frac{d}{60}$. So $d = 60 \cos 40^\circ \approx 45.96$.

16. a) Each number after $a_0 = 7$ is three move than the preceding number. Hence $a_n = 3n + 7$.

17. d) Place the rooks in columns from left to right. The rook in the left column can be placed in any of the 8 rows; the rook in the next column can be placed in any of the 7 remaining rows; etc. Thus the number of possibilities is $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$.

18. a) The minimum of this quadratic function occurs when $x = \frac{-16}{2 \cdot 4} = -2$. Thus the smallest value is $f(-2) = 16 - 32 + k = k - 16$.

19. d) $(\cos 2\alpha)(\sec^2 \alpha) = (\cos^2 \alpha - \sin^2 \alpha)\frac{1}{\cos^2 \alpha} = 1 - \frac{\sin^2 \alpha}{\cos^2 \alpha} = 1 - \tan^2 \alpha$.

20. b) Let x be the decimal. Then $100x - x - 137$. So $x = \frac{137}{99}$. This is in lowest terms, so $137 + 99 = 236$.

21. c) Let $z = x^2$. By the Remainder Theorem, dividing $z^5 - 1$ by $z - a$ gives a remainder of $a^5 - 1$.

22. b) If a pair of parallel sides of a trapezoid have the same length, the trapezoid is a parallelogram and the other pair of sides must be of the same length. Thus the side of length 1 must be parallel to a side of length 2. It follows that the trapezoid is isosceles, and the altitude is $\sqrt{2^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{15}}{2}$. Thus the area is $\frac{2+1}{2}\frac{\sqrt{15}}{2} = \frac{3}{4}\sqrt{15}$.

23. d) $\operatorname{Arctan} x = 45^\circ - \operatorname{Arctan} a$. So $x = \tan(45^\circ - \operatorname{Arctan} a) = \frac{\tan 45^\circ - \tan(\operatorname{Arctan} a)}{1 + \tan 45^\circ \tan(\operatorname{Arctan} a)} = \frac{1-a}{1+a}$.

24. d) We can say that $0 < bc < 10^{-1}$. Of the choices given, $|b| + |c| > 0$ is the only one that must hold.

25. d) The number of diagonals of a regular n-gon is $\binom{n}{2} - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$. We want $\frac{n(n-3)}{2} = 44$. So $n(n-3) = 88$, or $n = 11$.

26. b) Let r be their rate in still water. Time equals distance over rate, so $\frac{4}{r+3} + \frac{4}{r-3} = 1$. This gives $r^2 - 8r - 9 = 0$, so $r = 9$.

27. c) The choice of 4 of the 8 steps to be horizontal uniquely determines a path. Thus there are $\binom{8}{4} = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2} = 70$ different paths. for a path to go through $(2,2)$, two of the first four steps must be horizontal and two of the last four steps must be horizontal. There are $\binom{4}{2}^2 = 6^2 = 36$ such paths. Thus the probability is $\frac{36}{70} = \frac{18}{35}$.

28. e) Perfect squares end in the digits 1, 4, 9, 6, or 5.

29. c) The portion of the square inside the circle has area $\frac{\pi q^2}{4}$. Thus we want $\frac{\pi q^2}{4} = \frac{12^2}{2}$. So $q = 12\sqrt{\frac{2}{\pi}} \approx 9.575$.

30. b) The product of 2 through 21 involves the eight primes 2, 3, 5, 7, 11, 13, 17, and 19.

31. e) A can occur in 4 ways depending on the suit of the two. B can occur in 4 ways depending on the suit of the straight. C can occur in 4 ways depending on the suit of the missing king.

32. d) $\log_b 27 = 1.5$ means $b^{1.5} = 27$. So $b = 27^{2/3} = 9$.

33. e) The factors in some order must be -2, -1, 1, and 2. Thus the integers must be 5, 6, 8, and 9. Thus sum is 28.

34. b) The last two digits of the powers of 444 cycle through the pattern of ten numbers 44, 36, 84, 96, 24, 56, 64, 16, 04, and 76. Thus $444^{88} = (444^{10})^8(444^8)$ ends in 16.

35. b) We have $21x = 2000 + 1101 = 10101$. Long division in base 3 gives

$$
\begin{array}{r}
111 \\
21\,\overline{)\,10101} \\
\underline{21} \\
100 \\
\underline{21} \\
21 \\
\underline{21}
\end{array}
$$

36. c) Let the circle have radius 1. Then the altitude of the triangle is 3, and the base is $2\sqrt{3}$. The ratio of the areas is $\frac{\pi}{\frac{1}{2}(2\sqrt{3})(3)} \approx .6046$.

37. d) There are $4^3 = 64$ triples. Thus there are $\frac{64}{20} = 3.2$ times as many triples as amino acids.

38. e) Triangle ABC is equilateral. Its altitude has length $\frac{\sqrt{3}}{2}$. So the distance from a vertex to the center is $\frac{2}{3}\frac{\sqrt{3}}{2} = \frac{11}{\sqrt{3}} > \frac{11}{20}$. Thus D cannot be $\frac{11}{20}$ from A, B, and C.

39. e) a is the negative of the sum of the roots. $-b = -4\cdot3 - 4\cdot24 + 3\cdot24 = -36$ is the sum of products of pairs of the roots. And c is the negative of the product of the roots.

40. d) The train is moving for $12 - \frac{mn}{60}$ hours. The number of miles it travels is $40\left(12 - \frac{mn}{60}\right) = 480 - \frac{2mn}{3} = \frac{1440 - 2mn}{3}$.