Forty-Third Annual

Michigan Mathematics Prize Competition

Sponsored by the
Michigan Section of the Mathematical Association of America

Part II solutions

December 8, 1999

1. Based on the information given, we can fill in the following table.

team	*changed places with this many teams below it*	*changed places with this many teams above it*	*net change*	*1996 rank*	*1999 rank*
Michigan	0	0	0	1	1
Minnesota	0	0	0	2	2
Iowa	2	0	down 2	3	5
Indiana	6	0	down 6	4	10
Michigan State	1	2	up 1	5	4
Purdue	3	1	down 2	6	8
Northwestern	2	2	0	7	7
Ohio State	2	1	down 1	8	9
Penn State	0	6	up 6	9	3
Wisconsin	0	4	up 4	10	6

Thus the 1999 ranking is

1. Michigan
2. Minnesota
3. Penn State
4. Michigan State
5. Iowa
6. Wisconsin
7. Northwestern
8. Purdue
9. Ohio State
10. Indiana

2. a. For $t = \frac{\pi}{2}$ we have $A = A\sin\frac{\pi}{2} + B\cos\frac{\pi}{2} = C\sin\left(\frac{\pi}{2} + D\right) = C\cos D$.
For $t = 0$ we have $B = A\sin 0 + B\cos 0 = C\sin(0 + D) = C\sin D$.
Thus $A^2 + B^2 = C^2(\cos^2 D + \sin^2 D) = C^2$. Since $C > 0$, we have $C = \sqrt{A^2 + B^2}$.
Also, $\tan D = \frac{\sin D}{\cos D} = \frac{C\sin D}{C\cos D} = \frac{B}{A}$. So $D = \tan^{-1}\frac{B}{A}$.

b. Since $-\frac{\pi}{2} < D < \frac{\pi}{2}$, we know that $\cos D > 0$ and $\sec D > 0$. Using $\tan D = \frac{B}{A}$ and $C^2 = A^2 + B^2$, we find that

$$\cos D = \frac{1}{\sec D} = \frac{1}{\sqrt{\sec^2 D}} = \frac{1}{\sqrt{1 + \tan^2 D}} = \frac{1}{\sqrt{1 + \frac{B^2}{A^2}}} = \frac{A}{\sqrt{A^2 + B^2}} = \frac{A}{C}.$$

Thus,

$$\sin D = \pm\sqrt{1 - \cos^2 D} = \pm\sqrt{1 - \frac{A^2}{C^2}} = \frac{\pm\sqrt{C^2 - A^2}}{C} = \pm\frac{B}{C}.$$

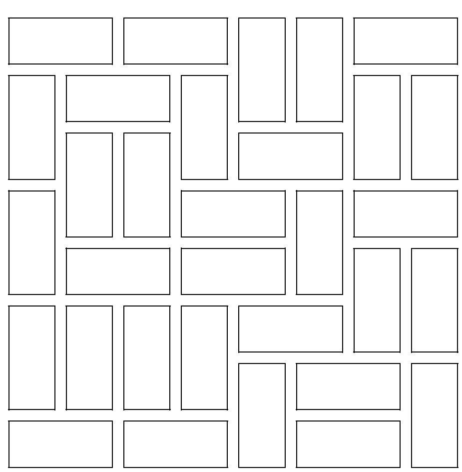
Now $\sin D$ has the same sign as $\tan D = \frac{B}{A}$. Since $C > 0$ and $A = C\cos D > 0$, we must have $\sin D = \frac{B}{C}$. Therefore

$$\begin{aligned} C\sin(t + D) &= C(\sin t\cos D + \cos t\sin D) \\ &= C(\sin t)\frac{A}{C} + C(\cos t)\frac{B}{C} \\ &= A\sin t + B\cos t. \end{aligned}$$

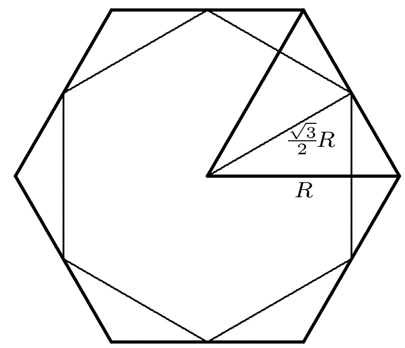
c. $3\sin t + 2\cos t = \sqrt{3^2 + 2^2}\,\sin(t + D) = \sqrt{13}\,\sin(t + D)$ has a maximum value of $\sqrt{13}$.

3. There are five horizontal lines and five vertical lines that can cut the board. Since the area on either side of such a line is even, each of these ten lines must cut an even number of dominoes. No domino is cut by more than one line. For each line to cut a domino (and hence cut at least one other domino), we would need at least 20 dominoes. Since there are only 18, some line must cut the board but not cut any of the dominoes.

Note: This result does not extend to an 8×8 board.

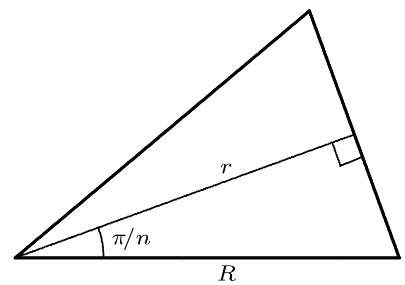


4. a. Let R be the distance from the center of the hexagon to a vertex. The distance from the center to the vertex of the new hexagon is the length of an altitude of an equilateral triangle of side length R. Thus this distance is $\frac{\sqrt{3}}{2}R$. The ratio of the area of the new hexagon to the are of the original hexagon is $\left(\frac{\frac{\sqrt{3}}{2}R}{R}\right)^2 = \frac{3}{4}$.



b. Let R be the distance from the center of the n-gon to a vertex. Let r be the distance from the center to the midpoint of a side. In the right triangle with vertices at the center, the midpoint of a side of the n-gon, and an endpoint of that side, we have $\cos\frac{\pi}{n} = \frac{r}{R}$. Thus the ratio of the areas of the two n-gons is

$$\left(\frac{r}{R}\right)^2 = \cos^2\left(\frac{\pi}{n}\right).$$



5. Take one key to each of the 90 rooms and give them to 90 of the guests, one key per person. Give each of the 10 remaining guests keys to all 90 rooms. This requires $90 + 10 \cdot 90 = 990$ keys. After dinner, the guests with one key first go to the room to which they have a key. Then any remaining guests can go to any of the remaining room.

With fewer that 990 keys, at least one room will have 10 or fewer keys to it given out. If the guests with these keys are among the 10 who do not stay, none of the guests that night will have a key to that room. Thus the solution given above requires the fewest keys.

What happens if you have to assign rooms as the guests arrive (without waiting for all ninety to arrive for dinner)?