Forty-Third Annual

Michigan Mathematics Prize Competition

Solutions to Part I

October 13, 1999

1. e) The volume of a solid increases as the cube of its linear dimension. Thus the ratio is 11^3.

2. b) The outcome in (i) is one of six possible orders of the three numbers that can occur in event (ii).

3. b) We need to find where $\frac{x-1}{x+1} - 2 = \frac{x-1-2(x+1)}{x+1} = \frac{-x-3}{x+1}$ is positive. The fraction changes sign only at $x = -3$ and $x = -1$, and it is negative for large values of x.

4. b) For $x < 0$, $\frac{1}{x}$ decreases and is negative, so $f\left(\frac{1}{x}\right)$ decreases, and thus $g(x)$ increases. For $x > 0$, $\frac{1}{x}$ decreases and is positive, so $f\left(\frac{1}{x}\right)$ increases, and thus $g(x)$ decreases.

5. c) For $x \neq -1$, the equation is equivalent to $x^2 + x - \alpha = 0$. The discriminant $1 + 4\alpha$ can be positive (giving two distinct roots for $\alpha > -\frac{1}{4}$, except that $N(0) = 1$), zero (giving one root for $\alpha = -\frac{1}{4}$), or negative (giving no real roots for $\alpha < -\frac{1}{4}$).

6. a) From the 15% who failed canoe-safety, we subtract the 7% who failed both tests, leaving 8% who passed swimming but not canoe-safety. This represents $\frac{8\%}{90\%} \approx 8.9\%$ of the 90% who passed the swimming test.

7. d) The set A must either consist of all the even numbers in S or all the odd numbers in S.

8. a) All points on the circle $x^2 + y^2 = 1$ satisfy $|x| \leq 1$ and $|y| \leq 1$. For $|x| \leq 1$, the points on the line satisfy $y = \frac{10-2x}{5} \geq \frac{8}{5} > 1$. [Answer c) was also accepted for students who considered complex solutions.]

9. d) $f(2i+1) = (2i+1)^2 - 2(2i+1) = -4 + 4i + 1 - 4i - 2 = -5$, and $f(-5) = (-5)^2 - 2(-5) = 25 + 10 = 35$.

10. a) $5 \cdot 1 - 2 \cdot 2 - 1 = 0$.

11. c) The information rules out the first of the four equally likely possibilities gg, gb, bg, bb for two children.

12. d) The period is $\frac{2\pi}{c}$, and B is $\frac{3}{4}$ of a period from A. Thus the x-coordinate of B is $\frac{1}{c} + \frac{3}{4} \cdot \frac{2\pi}{c} = \frac{1}{c} + \frac{3\pi}{2c}$.

13. c) At 5 minutes, the side length of the cube is 10 feet, so the surface area is $6 \cdot 10^2 = 600$. When the volume doubles, the side length increases by a factor of $2^{1/3}$, and the surface areas increases by the square of this factor.

14. a) Assume everyone shook hands with a different number of people. Since the handshakes are with other people, the number of possibilities are $0, 1, \ldots, n-1$. With n people all of these possibilities would have to be used. Thus one person shakes hands with 0 people. This leaves only $n-2$ people for the others to shake hands with. In particular, no one can shake hands with $n-1$ other people as required by the assumption.

15. b) The angle subtended by the arc AC has measure $40°$. Thus the ratio between the length of the circumference and the length of the arc AC is $\frac{360}{40} = 9$. It follows that the length of the circumference is $2 \cdot 9 = 18$, and hence the radius is $\frac{18}{2\pi} = \frac{9}{\pi} \approx 2.86$.

16. c) The inequality is equivalent to $-5 < 2x - 1 < 5$. So $-4 < 2x < 6$, and thus $-2 < x < 3$.

17. c) This is a geometric series with first term $\frac{1}{2} + \frac{1}{4}$ and ratio $\frac{1}{16}$. Thus the sum is $\frac{3/4}{1-1/16} = \frac{3}{4}\frac{16}{15} = \frac{4}{5}$.

18. a) The greatest common divisor of $321 = 3 \cdot 107$ and $123 = 3 \cdot 41$ is 3. Thus the insect will cross a 4-corner intersection only at $(107, 41)$ and $(214, 82)$.

19. b) $x + 1$ is a factor if and only if $x = -1$ is a zero. Now $(-1)^n - (-1) - 2 = (-1)^n - 1 = 0$ if and only if n is even.

20. c) $3\left(z + \frac{1}{z}\right) = \left(z + \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right) = \left(z + \frac{1}{z}\right)^3 = z^3 + 3z^2\frac{1}{z} + 3z\frac{1}{z^2} + \frac{1}{z^3} = z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)$. Thus $z^3 + \frac{1}{z^3} = 0$.

21. b) To complete the flush, you need to draw one of the remaining 9 diamonds from the remaining 48 cards. Thus the probability is $\frac{9}{48} = \frac{3}{16}$.

22. d) Volume is proportional the cube of the radius, and surface area is proportional to the square of the radius. Thus the ratio is proportional to the radius.

23. c) In the second quadrant, $y = 10^x$ is asymptotic to the negative x-axis and increases to a y-intercept of 1, whereas $y = x^{10}$ is positive and decreases to a y-intercept of 0. Thus there is exactly one negative solution ($x \approx -0.82667$). There is an obvious solution at $x = 10$. But since the exponential function grows more rapidly than the quadratic function, there must be another positive solution ($x = 1.37129$).

24. e) The desired score x on the final exam satisfies $(.6)(.7) + .4x = .8$. Thus $x = \frac{.8-.42}{.4} = .95$.

25. b) The 1 must occur in one of the first $n-1$ positions, and the 2 must occur in the one position (out of the remaining $n-1$) to the right of the 1. The probability is $\frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$.

26. c) The last two digits of a_n depend only on the last two digits of a_{n-1}. These digits are 03, 12, 56, 92, 56, 92 and then alternating 56 and 92. The odd term a_{1999} will end in 56.

27. a) $\frac{x}{2} + \frac{1}{x} - \sqrt{2} = \left(\sqrt{\frac{x}{2}} - \frac{1}{\sqrt{x}}\right)^2 \geq 0$ with equality when $x = \sqrt{2}$.

28. a) At the beginning the water will touch the three sides that meet at the corner on the table. After the water reaches another corner of these sides, it will touch the other three sides of the cube. When it rises above the top corner of the original sides, the air above the water will touch only the three upper sides of the cube.

29. e) The first factor changes sign at $x = -1$ and at $x = 1$; the second factor changes sign at $x = -1$; and the product is negative for large values of x. Thus as x decrease, the expression becomes positive as x decreases through 1, it goes to 0 at $x = -1$, and becomes positive again for x less than -1.

30. b) After the first card is drawn, out of the 51 remaining cards, the second card must be one of the 12 remaining card in the same suit as the first card. Thus the probability is $\frac{12}{51} = \frac{4}{17}$.

31. e) With seven marks, at least one of the vertical columns must have three marks. Six marks can be placed off one of the diagonals without having three in a row.

32. a) The complementary angle is $90 - (50 - x) = 40 + x$ in degrees. This is $\frac{\pi}{180}(40 + x)$ radians.

33. b) We know $p(1) = 3$, $p(3) = 5$, and $p(x) = q(x)(x-1)(x-3) + ax + b$. Thus $3 = p(1) = q(1) \cdot 0 + a + b = a + b$, and $5 = p(3) = q(3) \cdot 0 + 3a + b = 3a + b$. It follows that $a = 1$ and $b = 2$.

34. b) $2(2n + 5) = 5n + 2$. So $n = 8$.

35. a) The length of a side of the square is equal to the length of the diameter of the inscribed circle. The diagonal of the square is a diameter of the circumscribed circle. The diagonal is $\sqrt{2}$ times the length of the side. The area of a circle is proportional to the square of the diameter.

36. c) $x = y^2 - y + 2 = \left(y^2 - y + \frac{1}{4}\right) + \frac{7}{4} = \left(y - \frac{1}{2}\right)^2 + \frac{7}{4}$. So $\left(\frac{7}{4}, \frac{1}{2}\right)$ is the vertex of the parabola that opens to the right.

37. a) The graph of $y = (x-a)(x-b)$ is parabola opening upward and crossing the x-axis at a and b.

38. a) For food costing x dollars, you will pay $x + .05x + .15x = 1.2x$. If $1.2x \leq 18$, then $x \leq 15$.

39. a) The complex number $1 - i$ has angle $-\frac{\pi}{4}$ and modulus $\sqrt{2}$. Thus its tenth power has angle $10\left(-\frac{\pi}{4}\right) = -\frac{5\pi}{2}$ and modulus $\left(\sqrt{2}\right)^{10} = 32$. The number on the negative imaginary axis at a distance of 32 from the origin is $-32i$.

40. c) The lateral surface area (in square centimeters) of a 1-meter section of pipe of diameter d is $100(\pi d)$. The ribbon covers 1500 square centimeters. Equating these two expression gives $d = \frac{1500}{100\pi} \approx 4.77$.