42nd Annual MMPC
PART II Answer Key

1. a) Look at $N \leq 61$. $50 N = 500 + 40 N$ implies $N = 50$. Since the costs for the first caterer increase more steeply than the costs for the second catere, the second caterer is less expensive for $N > 50$.

b) Price for N = 50 is \$2500 so $2500 \log(x/4) = 5000$ implies that x = 400.

c) Profit = $400 \cdot 60 - 500$ = \$19,000

2. a) N = 4k = (2k-1) + (2k+1)

b) Suppose N = x + (x+2) +   + (x+2k) for x odd and $k \geq 1$.

Then N = x (k+1) + 2(1+2+   +k) = x (k+1) + 2(1/2)k(k+1) = (x+k) (k+1) and so N is not prime.

c) Suppose N = x + (x+2) +   + (x+2k) = (x+k) (k+1) where x is odd. If k is odd then x+k and k+1 are even and N cannot be twice an odd integer. If k is even then x+k and k+1 are both odd so N cannot be twice an odd integer.

3. a) $\sum \frac{1}{(2n)^2} = \sum \frac{1}{4n^2} = \frac{1}{4}\sum \frac{1}{n^2} = \frac{1}{4}S$

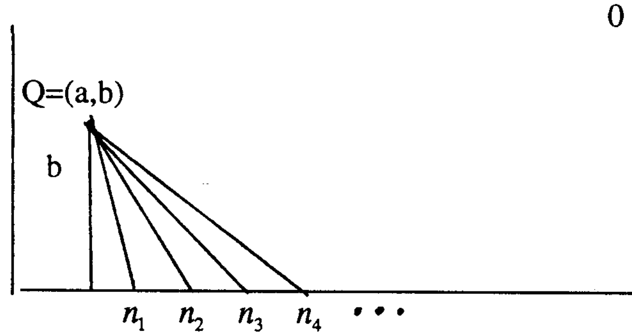
b) $\sum \frac{1}{(2n-1)^2} = S - \sum \frac{1}{(2n)^2} = S - \frac{1}{4}S = \frac{3}{4}S$

c) (part b) – (part a) = (1/2) S

4. a)

$P_i = (n_i, 0)$

$0 < n_1 < n_2 < n_3 < \cdots$



Can assume that $0 \leq a < n_1$. Suppose that there is an infinite number of I's such that the distance $z_i = |P_i Q|$ is an integer. Then for these I's $b^2 + n_i^2 \geq z_i^2$ so $z_i^2 - n_i^2 \leq b^2$

Since n_i and z_i are both integers, $b^2 \geq z_i^2 - n_i^2 \geq 2n_i + 1$ but n_i's get arbitrarily large since they are an infinite number of distinct integers. Contradiction

5. a) 2 Area Δ ABC = x AB + y BC + z AC = (x+y+z) AB since the triangle is equilateral. Thus x+y+z = 2 Area ΔABC / AB = constant

b) Since P is on a line parallel to AB x is constant and since CA = CB because the triangle is isosceles we have y BC + z AC = 2 Area ΔABC – x AB so y+z = (2 Area Δ ABC – x AB)/ AC = constant and thus x+y+z = constant.

c) Let AB be the longest side and AC the shortest side. Let F(P) = x+y+z .
If h_A = length of the altitude from A etc. then F(A) = h_A and the same for F(B) and F(C).
Since AB > BC > AC we have $h_C < h_A < h_B$. Pick Q on BC such that F(Q) = h_A = F(A)
.
Then F(P) is constant on the line AQ. This can be seen by using similar triangles or noting that F(P) is a linear function.

This problem can be solved using analytic geometry. Put in a ut-coordinate system with A = (0,0), B = (c, 0) and C = (a, b). x+y+z can then be computed using equations of lines and distance formulas and m can be found such that x+y+z is constant on the line u = mt