Forty-Second Annual

Michigan Mathematics Prize Competition

Sponsored by the

Michigan Section of the Mathematical Association of America

Part II

December 9, 1998

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1. An organization decides to raise funds by holding a $60 a plate dinner. They get prices from two caterers. The first caterer charges $50 a plate. The second caterer charges according to the following schedule: $500 set-up fee plus $40 a plate for up to and including 61 plates, and $2500 \log_{10}(\frac{p}{4})$ for $p > 61$ plates.
   a) For what number of plates $N$ does it become at least as cheap to use the second caterer as the first?
   b) Let $N$ be the number you found in a). For what number of plates $X$ is the second caterer’s price exactly double the price for $N$ plates?
   c) Let $X$ be the number you found in b). When $X$ people appear for the dinner, how much profit does the organization raise for itself by using the second caterer?

2. Let $N$ be a positive integer. Prove the following:
   a) If $N$ is divisible by 4, then $N$ can be expressed as the sum of two or more consecutive odd integers.
   b) If $N$ is a prime number, then $N$ cannot be expressed as the sum of two or more consecutive odd integers.
   c) If $N$ is twice some odd integer, then $N$ cannot be expressed as the sum of two or more consecutive odd integers.

3. Let $S = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$.
   a) Find, in terms of $S$, the value of $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots$.
   b) Find, in terms of $S$, the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$.
   c) Find, in terms of $S$, the value of $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$. 
4. Let \( \{P_1, P_2, P_3, \ldots \} \) be an infinite set of points on the \( x \)-axis having positive integer coordinates, and let \( Q \) be an arbitrary point in the plane not on the \( x \)-axis. Prove that infinitely many of the distances \( |P_iQ| \) are not integers.
   a) Draw a relevant picture.
   b) Provide a proof.

5. Point \( P \) is an arbitrary point inside triangle \( ABC \). Points \( X \), \( Y \), and \( Z \) are constructed to make segments \( PX \), \( PY \), and \( PZ \) perpendicular to \( AB \), \( BC \), and \( CA \), respectively. Let \( x \), \( y \), and \( z \) denote the lengths of the segments \( PX \), \( PY \), and \( PZ \), respectively.
   a) If triangle \( ABC \) is an equilateral triangle, prove that \( x + y + z \) does not change regardless of the location of \( P \) inside triangle \( ABC \).
   b) If triangle \( ABC \) is an isosceles triangle with \( |BC| = |CA| \), prove that \( x + y + z \) does not change when \( P \) moves along a line parallel to \( AB \).
   c) Now suppose that triangle \( ABC \) is scalene (i.e., \( |AB|, |BC|, \) and \( |CA| \) are all different). Prove that there exists a line for which \( x + y + z \) does not change when \( P \) moves along this line.