Forty-Second Annual

Michigan Mathematics Prize Competition

Sponsored by the

Michigan Section of the Mathematical Association of America

Part II

December 9, 1998

1. An organization decides to raise funds by holding a \$60 a plate dinner. They get prices from two caterers. The first caterer charges \$50 a plate. The second caterer charges according to the following schedule: \$500 set-up fee plus \$40 a plate for up to and including 61 plates, and $\$2500\log_{10}\left(\frac{p}{4}\right)$ for $p > 61$ plates.

   a) For what number of plates N does it become at least as cheap to use the second caterer as the first?

   b) Let N be the number you found in a). For what number of plates X is the second caterer's price exactly double the price for N plates?

   c) Let X be the number you found in b). When X people appear for the dinner, how much profit does the organization raise for itself by using the second caterer?

2. Let N be a positive integer. Prove the following:

   a) If N is divisible by 4, then N can be expressed as the sum of two or more consecutive odd integers.

   b) If N is a prime number, then N cannot be expressed as the sum of two or more consecutive odd integers.

   c) If N is twice some odd integer, then N cannot be expressed as the sum of two or more consecutive odd integers.

3. Let $S = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$.

   a) Find, in terms of S, the value of $\dfrac{1}{2^2} + \dfrac{1}{4^2} + \dfrac{1}{6^2} + \dfrac{1}{8^2} + \cdots$.

   b) Find, in terms of S, the value of $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \cdots$.

   c) Find, in terms of S, the value of $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots$.

4. Let $\{P_1, P_2, P_3, \ldots\}$ be an infinite set of points on the x-axis having positive integer coordinates, and let Q be an arbitrary point in the plane not on the x-axis. Prove that infinitely many of the distances $|P_iQ|$ are not integers.
   a) Draw a relevant picture.
   b) Provide a proof.

5. Point P is an arbitrary point inside triangle ABC. Points X, Y, and Z are constructed to make segments PX, PY, and PZ perpendicular to AB, BC, and CA, respectively. Let x, y, and z denote the lengths of the segments PX, PY, and PZ, respectively.
   a) If triangle ABC is an equilateral triangle, prove that $x + y + z$ does not change regardless of the location of P inside triangle ABC.
   b) If triangle ABC is an isosceles triangle with $|BC| = |CA|$, prove that $x + y + z$ does not change when P moves along a line parallel to AB.
   c) Now suppose that triangle ABC is scalene (i.e., $|AB|$, $|BC|$, and $|CA|$ are all different). Prove that there exists a line for which $x + y + z$ does not change when P moves along this line.