SOLUTIONS TO PART II MMPC 1997

1. a) $\frac{4(3)^3}{3} = 36$ b) $\frac{5}{3}\left[4^3 - 2^3\right] = 93\frac{1}{3}$

   c) The boundary of the described region is a locus of points equidistant from the point (0,1) and the line $y = -1$. This is the parabola $y = 0.25\,x^2$. The area between this parabola and the x-axis on the interval [-1,1] is one-sixth of a square unit. Thus the probability of a dart landing on the described region is 11/24.

2. a) $\begin{vmatrix} 1 & 3 & -1 \\ 4 & 7 & 2 \\ 3 & -6 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -5 & 6 \\ 0 & -15 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & -10 \end{vmatrix} = 50$

   b) $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & -2 & -6 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & -6 & -12 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & 0 & -3 \end{vmatrix} = -12$

3. a) 32 b) 2^n c) $64 = 2(1 + 15 + 15 + 1)$

   d) Because $0 = [1 + (-1)]^n =$ (sum of all odd position entries) - (sum of all even position entries), the odd position sum is exactly half the sum of all entries in the row.

4. a) A cancellation pattern emerges, showing the answer to be $(1\cdot 2)/(11\cdot 12) = 1/66$.

   b) Note that $(k - 2)^2 + 14(k - 2) + 41 = k^2 + 10k + 17$, so the numerator terms trail the denominator terms by two. The resulting cancellation gives a result of $(28\cdot 41)/(248\cdot 241) = 287/17422$.

   c) Writing top and bottom in factored form leads to the discovery of a cancellation pattern in which the only uncancelled items are a 2 on top and a 3 on the bottom.

5. a) Triangle ABE is isosceles, so AE = 1. Angle AED is 67.5°, so angle AEF is 112.5°. The law of sines shows side FE has length $\sqrt{2}\sin 22.5°$, and triangle AFE has altitude $\cos 22.5°$. Thus its area is $\frac{1}{2}\sqrt{2}\sin 22.5° \cos 22.5° = \frac{\sqrt{2}}{4}\sin 45° = \frac{1}{4}$.

   b) Construct the circumcircle of triangle CAB and extend the median, angle bisector, and altitude to points G, H, and J. Because the angles at A are all equal, arcs CG, GH, HJ, and JB have equal length. Thus chord GJ is parallel to chord CB, and triangle AJG is a right triangle, implying that chord AFG is a diameter. The equality of arcs BH and HC and of segments CF and FB gives that segment FH, possibly extended, passes through O, the center of the circle, making segment HOK a diameter. But diameters AFG and HOK meet at F, which is then the center of the circle. Thus BC is a diameter and angle A is a right angle.