MMPC CODE NUMBER

FORTYFIRST ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 10, 1997

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page, clearly write the **problem number**. If you use additional paper for your answer, **check the appropriate box** and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. If you are unable to solve a particular problem, partial credit will be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

7. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference materials, computational aids, or any other aids is likewise prohibited. Please note that calculators are <u>**not**</u> allowed on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. It is not necessary to return scratch paper on which routine numerical calculations have been made.

8. You may now open the test booklet.

Problem	Score
1	
2	
3	
4	
5	
TOTAL	

1. It can be shown in Calculus that the area between the x-axis and the parabola

$y = kx^2$ (k is a positive constant) on the x-interval $0 \le x \le a$ is $\frac{ka^3}{3}$.

a) (1 pt.) Find the area between the parabola $y = 4x^2$ and the x-axis for $0 \le x \le 3$.

b) (2 pts.) Find the area between the parabola $y = 5x^2$ and the x-axis for $-2 \le x \le 4$.

c) (7 pts.) A square 2 by 2 dartboard is situated in the xy-plane with its center at the origin and its sides parallel to the coordinate axes. Darts that are thrown land randomly on the dartboard. Find the probability that a dart will land at a point of the dartboard that is nearer to the point (0, 1) than to the bottom edge of the dartboard.

Check this box if your solution is continued on inside back cover (page 7) ☐
Check this box if your solution is continued on inserted additional paper ☐

2. When two rows of a determinant are interchanged, the value of the determinant changes sign. There are also certain operations which can be performed on a determinant which leave its value unchanged. Two such operations are changing any row by adding a constant multiple of another row to it, and changing any column by adding a constant multiple of another column to it. Often these operations are used to generate lots of zeroes in a determinant in order to simplify computations. In fact, if we can generate zeroes everywhere below the main diagonal in a determinant, the value of the determinant is just the product of all the entries on that main diagonal. For example, given the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 6 & 2 \\ 3 & 10 & 4 \end{vmatrix}$, we add -2 times the first row to the second row, then add -2 times the second row to the third row, giving the new determinant $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{vmatrix}$, and the value is the product of the diagonal entries: 6.

a) (4 pts.) Transform this determinant into another determinant with zeroes everywhere below the main diagonal, and find its value: $\begin{vmatrix} 1 & 3 & -1 \\ 4 & 7 & 2 \\ 3 & -6 & 5 \end{vmatrix}$

b) (6 pts.) Do the same for this determinant: $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix}$

Check this box if your solution is continued on inside back cover (page 7) ☐
Check this box if your solution is continued on inserted additional paper ☐

3. In Pascal's triangle, the entries at the ends of each row are both 1, and otherwise each entry is the sum of the two entries diagonally above it:

Row Number									
0					1				
1				1		1			
2			1		2		1		
3		1		3		3		1	
4	1		4		6		4		1
....					....				

This triangle gives the binomial coefficients in expansions like $(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$.

a) (1 pt.) What is the sum of the numbers in row #5 of Pascal's triangle?

b) (2 pts.) What is the sum of the numbers in row #n of Pascal's triangle?

c) (2 pts.) Show that in row #6 of Pascal's triangle, the sum of all the numbers is exactly twice the sum of the first, third, fifth, and seventh numbers in the row.

d) (5 pts.) Prove that in row #n of Pascal's triangle, the sum of all the numbers is exactly twice the sum of the numbers in the odd positions of that row.

Check this box if your solution is continued on inside back cover (page 7) ❑

Check this box if your solution is continued on inserted additional paper ❑

4. The product of several terms is sometimes described using the symbol $\prod$, which is capital pi, the Greek equivalent of p, for the word "product". For example the symbol $\prod_{k=1}^{4}(2k+1)$ means the product of numbers of the form (2k + 1), for k = 1,2,3,4. Thus it equals 945.

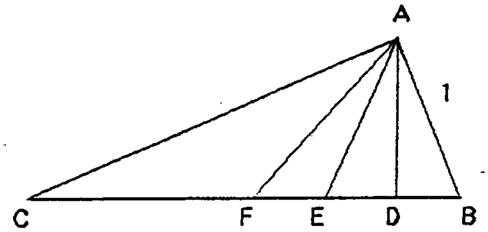
a) (2 pts.) Evaluate as a reduced fraction $\prod_{k=1}^{10}\frac{k}{k+2}$

b) (3 pts.) Evaluate as a reduced fraction $\prod_{k=1}^{10}\frac{k^2+10k+17}{k^2+14k+41}$

c) (5 pts.) Evaluate as a reduced fraction $\prod_{k=2}^{\infty}\frac{k^3-1}{k^3+1}$

Check this box if your solution is continued on inside back cover (page 7) ❑
Check this box if your solution is continued on inserted additional paper ❑

5. a) (3 pts.) In right triangle CAB, the median AF, the angle bisector AE, and the altitude AD divide the right angle A into four equal angles. If AB = 1, find the area of triangle AFE.



b) (7 pts.) If in any triangle, an angle is divided into four equal angles by the median, angle bisector, and altitude drawn from that angle, prove that the angle must be a right angle.

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Continuation of solution to problem number _______

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