40TH ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Part II Answer Key

1. a) (2 pts.) $\frac{1}{7} < \frac{2}{13} < \frac{1}{6}$

$$\therefore \frac{2}{13} = \frac{1}{7} + \left(\frac{2}{13} - \frac{1}{7}\right) = \frac{1}{7} + \left(\frac{14}{91} - \frac{13}{91}\right) = \frac{1}{7} + \frac{1}{91}$$

b) (2 pts) $\frac{1}{2} < \frac{9}{10} < 1$

$$\therefore \frac{9}{10} = \frac{1}{2} + \left(\frac{9}{10} - \frac{1}{2}\right) = \frac{1}{2} + \frac{2}{5}$$

now $\frac{1}{3} < \frac{2}{5} < \frac{1}{2}$

$$\therefore \frac{9}{10} = \frac{1}{2} + \frac{1}{3} + \left(\frac{2}{5} - \frac{1}{3}\right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{15}$$

c) (2 pts) $\frac{1}{k+1} < \frac{2}{2k+1} < \frac{1}{k}$

$$\therefore \frac{2}{2k+1} = \frac{1}{k+1} + \left(\frac{2}{2k+1} - \frac{1}{k+1}\right) = \frac{1}{k+1} + \frac{(2k+2)-(2k+1)}{(k+1)(2k+1)}$$

$$= \frac{1}{k+1} + \frac{1}{(k+1)(2k+1)}$$

d) (4 pts) $\frac{1}{2k+1} < \frac{3}{6k+1} < \frac{1}{2k}$

$$\therefore \frac{3}{6k+1} = \frac{1}{2k+1} + \left(\frac{3}{6k+1} - \frac{1}{2k+1}\right) = \frac{1}{2k+1} + \frac{2}{(6k+1)(2k+1)}$$

$$= \frac{1}{2k+1} + \frac{2}{12k^2+8k+1}$$

now $\frac{1}{6k^2+4k+1} < \frac{2}{12k^2+8k+1} < \frac{1}{6k^2+4k}$

$$\therefore \frac{3}{6k+1} = \frac{1}{2k+1} + \frac{1}{6k^2+4k+1} + \left(\frac{2}{12k^2+8k+1} - \frac{1}{6k^2+4k+1}\right)$$

$$= \frac{1}{2k+1} + \frac{1}{6k^2+4k+1} + \frac{1}{(12k^2+8k+1)(6k^2+4k+1)}$$

$$= \frac{1}{2k+1} + \frac{1}{6k^2+4k+1} + \frac{1}{(6k+1)(2k+1)(6k^2+4k+1)}$$

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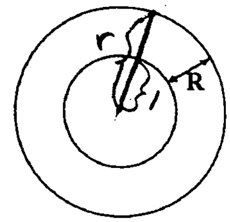
2. a) (2 pts) Let A_{sc} represent the area of the small circle, A_{Lc} represent the area of the large circle, and let r represent the radious of the large circle. Then, because the small circle has a radius of one unit, $A_{sc} = \pi$. Also, $A_{Lc} = \pi r^2$ and $2A_{sc} = A_{Lc}$. Thus, $2\pi = \pi r^2$, $r = \sqrt{2}$, and $R = \sqrt{2} - 1$.



b) (8 pts.) Let A_S represent the area of the small triangle and let A_L represent the area of the larger triangle. Let s represent the length of a side of the larger equilateral triangle and let h represent the height of the larger triangle. Then, because the small equilateral triangle has sides of length one unit and a height of $\frac{\sqrt{3}}{2}$ units, $A_S = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$.

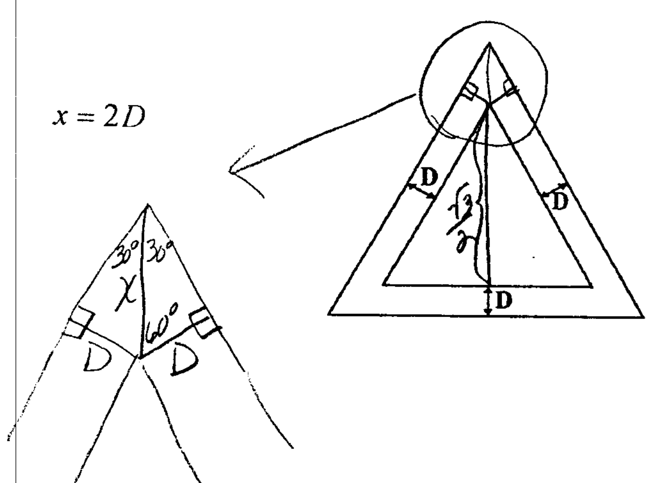
In the larger triangle, $h = \frac{\sqrt{3}}{2}s$, so $A_L = \frac{1}{2}hs = \frac{1}{2}\left(\frac{\sqrt{3}}{2}s\right)s = \frac{\sqrt{3}}{4}s^2$. But,

$2A_S = A_L$, so $2\left(\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{2}s^2 \Rightarrow s^2 = 2 \Rightarrow s = \sqrt{2}$ and $h = \left(\frac{\sqrt{3}}{2}\right)\sqrt{2} = \frac{\sqrt{6}}{2}$.

However, (see figure) $h = \frac{\sqrt{3}}{2} + x + D = \frac{\sqrt{3}}{2} + 2D + D = \frac{\sqrt{3}}{2} + 3D$. Setting the two expressions for h equal to each other,

$$\frac{\sqrt{6}}{2} = \frac{\sqrt{3}}{2} + 3D \Rightarrow 3D = \frac{\sqrt{6}-\sqrt{3}}{2} \Rightarrow D = \frac{\sqrt{6}-\sqrt{3}}{6} = \frac{\sqrt{3}(\sqrt{2}-1)}{6}.$$

$x = 2D$



c) (4 pts.) The top-most ball, and the middle ball on the right side of the $3x3$ square, are, by symmetry, both tangent to the centerline (altitude) of the triangle that forms the right face of the pyramid. If we call (0,0,0) the center of the top ball and $(2,0,-2\sqrt{2})$ the center of the other ball, the great circles in the plane $y = 0$ have the centerline of the triangular face as a common tangent.

That common tangent has slope $\frac{\Delta z}{\Delta x}$, the slope of the line joining their centers, i.e., $\frac{-2\sqrt{2}}{2} = -\sqrt{2}$.

The radius through $z = 0$ in the plane $y = 0$ has slope $\frac{\Delta z}{\Delta x} = \frac{-1}{-\sqrt{2}} = \frac{1}{\sqrt{2}}$. Thus, the point of tangency satisfies $z = \frac{x}{\sqrt{2}}$ and $x^2 + z^2 = 1$.

$$\therefore x^2 + \frac{1}{2}x^2 = 1$$

$$\frac{3}{2}x^2 = 1$$

$$x = \sqrt{\frac{2}{3}}, \quad z = \frac{x}{\sqrt{2}} = \sqrt{\frac{1}{3}}$$

The altitude (centerline) of the triangular face has equation (in plane $y = 0$) of

$$z - \sqrt{\frac{1}{3}} = -\sqrt{2}\left(x - \sqrt{\frac{2}{3}}\right) = -\sqrt{2}x + \frac{2}{\sqrt{3}}.$$

$$\therefore z = -\sqrt{2}x + \frac{3}{\sqrt{3}} = -\sqrt{2x} + \sqrt{3}.$$

Thus, the z-intercept (the peak of the pyramid) is $\sqrt{3}\,m$ above the center of the highest ball. So the height for the pyramid is $c = 9\sqrt{2} + 1 + \sqrt{3}$ meters. Now if we re-coordinatize the plane $y = 0$, so that the origin is at the center of the square, then the centerline of the triangular face still has slope $= -\sqrt{2}$ and z-intercept c. So, it has equation $z = -\sqrt{2}x + c$. This intersects the x-axis at $\frac{c}{\sqrt{2}}$. Since x represents the distance from the center of the square base to the middle of the right hand edge, each side of the square base has length $2\left(\frac{c}{\sqrt{2}}\right) = \sqrt{2}c$ meters. The volume of the pyramid is therefore $\frac{1}{3}\left(\sqrt{2}c\right)^2(c) = \frac{2}{3}c^3$ cubic meters (where $c = 9\sqrt{2} + 1 + \sqrt{3}$).

3. a) (1 pt.) Suppose $f(x) = ax + b$ has a fixed point. Then $\exists x_0$ such that $x_0 = ax_0 + b$ or $0 = (a-1)x_0 + b$ which has the solution $x_0 = \frac{-b}{a-1}$ as long as $a \neq 1$. If $a = 1$ and $b = 0$, then every point of f is a fixed point; but if $a = 1$ and $b \neq 0$, then f has no fixed point.

b) (1 pt.) If $a \neq 1$, $x_0 = \frac{-b}{a-1}$ is a fixed point.

c) If there is a fixed point of f_c, then $x_0^2 - c = x_0$ or $x_0^2 - x_0 - c = 0$ which has one solution when $(-1)^2 - 4(1)(-c) = 1 + 4c = 0$;
has two solutions when $1 + 4c > 0$;
has no solutions when $1 + 4c < 0$.

$\therefore$ i) (2 pts.) When $1 + 4c > 0$ or $c > \frac{-1}{4}$, there are two fixed points.

ii) (2 pts.) When $1 + 4c < 0$ or $c < \frac{-1}{4}$, there are no fixed points.

iii) (2 pts.) $x_0 = \frac{1 \pm \sqrt{1+4c}}{2}$

$\therefore \frac{1}{2} + \frac{1}{2}\sqrt{1+4c}$ and $\frac{1}{2} - \frac{1}{2}\sqrt{1+4c}$ are the fixed points.

d) There are many examples, e.g., $f(x) = x^3$ and $f(x) = k \sin x$ $(1<k<\pi)$.

4. a) (2 pts.) It is easiest to note that the sum of the first n squares is $\frac{n(n+1)(2n+1)}{6}$ which equals 385 when $n = 10$.

b) (4 pts.) Consider any ball (with center A) and the 4 supporting balls (with centers B_1, B_2, B_3, B_4). These 5 centers form a pyramid with a square base. One side of the square base is 2 meters. The center A is 2 meters from each of the B_i since the sphere is tangent to each of the 4 balls. If we let C be the centroid of this square, then the distance from C to each B_i is half of the diagonal or $\sqrt{2}$ meters. Now $\forall(i = 1,2,3,4) ACB_i$ is a right triangle with a hypotenuse of length $2m$ and one leg of length $\sqrt{2}\,m$. By the Pythagorean Theorem, the other leg must also have length $\sqrt{2}\,m$. Thus the centers of each "level" of balls is $\sqrt{2}\,m$ above the "level" of the centers of the supporting balls. The floor is $1m$ below the lowest level of centers and the top of the highest ball is $1m$ above the highest level. There are 10 levels, so the "height" of the pyramid (from the floor to the top of the highest ball is $2 + 9\sqrt{2}$ *meters*.

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5. a) (2 pts.) There are $C(21,3) = 1330$ possible 3 card hands. There are $(5)(3)(3)(3) = 135$ possible "straights" (including straight flushes which were not mentioned). The probability is $\frac{135}{1330} = \frac{27}{266}$ that the hand will contain a straight.

b) (2 pts.) There are 1330 possible hands and $3xC(7,3) = 105$ possible flushes (including the 15 straight flushes). The probability is therefore $\frac{105}{1330} = \frac{3}{38}$ that the hand will contain a flush.

c) (3 pts.) There are now $C(22,3)= 1540$ possible hands. There are 105 hands (see b) that are flushes without a joker and $3xC(7,2)=63$ hands containing a joker that are flushes. The probability is therefore $\frac{105+63}{1540} = \frac{168}{1540} = \frac{6}{55}$ that a 3 card hand will be a flush.

d) (3 pts.) Let s equal the number of suits and d equal the number of denominations. We need to find s and d so that the number of possible straights is equal to the number of possible flushes. The number of straights without a joker is $(d-2)(s^3)$. The number of straights with a joker must be counted by considering those of the form *1,2,joker*; *2,3,joker*; ...; *(d-1),d,joker* (and there are $(d-1)s^2$ of those) plus those of the form *1,3,joker*; *2,4,joker*; ... ; *(d-2),d,joker* (and there are $(d-2)s^2$ of those). So there are a total of

$$(d-2)s^3 + (2d-3)s^2 \text{ possible straights.}$$

There are $s\left(\frac{d(d-1)(d-2)}{6}\right)$ possible flushes without a joker and $s\left(\frac{d(d-1)}{2}\right)$ possible flushes with a joker. This gives a total of

$$s\left(\frac{d(d-1)(d-2)}{6}\right) + s\left(\frac{d(d-1)}{2}\right) = (s/6)d(d-1)(d+1) \text{ or}$$

$$(s/6)d(d^2-1) \text{ possible flushes.}$$

We are assuming $s>1$ and $d>3$. We need to find an integer solution to the equation: $(d-2)s^3 + (2d-3)s^2=(s/6)d(d^2-1)$ or (factoring an s out of both sides)

$$(d-2)s^2 + (2d-3)s=[d(d^2-1)]/6.$$

Proceeding by trial and error, when $d = 11$, this becomes

$$9s^2+19s=220$$

which has a solution $s = 4$. So a deck consisting of 4 suits and 11 denominations has an equal likelihood of producing a straight and a flush in a 3-card hand.