1. a) (2 pts.) \[ \frac{1}{7} < \frac{2}{13} < \frac{1}{6} \]
   \[ \therefore \frac{2}{13} = \frac{1}{7} + \left( \frac{2}{13} - \frac{1}{7} \right) = \frac{1}{7} + \left( \frac{14}{91} - \frac{13}{91} \right) = \frac{1}{7} + \frac{1}{91} \]

   b) (2 pts.) \[ \frac{1}{2} < \frac{9}{10} < 1 \]
   \[ \therefore \frac{9}{10} = \frac{1}{2} + \left( \frac{9}{10} - \frac{1}{2} \right) = \frac{1}{2} + \frac{2}{5} \]
   now \[ \frac{1}{3} < \frac{2}{5} < \frac{1}{2} \]
   \[ \therefore \frac{9}{10} = \frac{1}{2} + \frac{1}{3} + \left( \frac{2}{5} - \frac{1}{3} \right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{15} \]

   c) (2 pts.) \[ \frac{1}{k+1} < \frac{2}{2k+1} < \frac{1}{k} \]
   \[ \therefore \frac{2}{2k+1} = \frac{1}{k+1} + \left( \frac{2}{2k+1} - \frac{1}{k+1} \right) = \frac{1}{k+1} + \frac{(2k+2) - (2k+1)}{(k+1)(2k+1)} \]
   \[ = \frac{1}{k+1} + \frac{1}{(k+1)(2k+1)} \]

   d) (4 pts.) \[ \frac{1}{2k+1} < \frac{3}{6k+1} < \frac{1}{2k} \]
   \[ \therefore \frac{3}{6k+1} = \frac{1}{2k+1} + \left( \frac{3}{6k+1} - \frac{1}{2k+1} \right) = \frac{1}{2k+1} + \frac{2}{(6k+1)(2k+1)} \]
   \[ = \frac{1}{2k+1} + \frac{2}{12k^2 + 8k + 1} \]
   now \[ \frac{1}{6k^2 + 4k + 1} < \frac{2}{12k^2 + 8k + 1} < \frac{1}{6k^2 + 4k} \]
   \[ \therefore \frac{3}{6k+1} = \frac{1}{2k+1} + \frac{1}{6k^2 + 4k + 1} + \left( \frac{2}{12k^2 + 8k + 1} - \frac{1}{6k^2 + 4k + 1} \right) \]
   \[ = \frac{1}{2k+1} + \frac{1}{6k^2 + 4k + 1} + \frac{1}{(12k^2 + 8k + 1)(6k^2 + 4k + 1)} \]
   \[ = \frac{1}{2k+1} + \frac{1}{6k^2 + 4k + 1} + \frac{1}{(6k+1)(2k+1)(6k^2 + 4k + 1)} \]
2. a) (2 pts) Let $A_s$ represent the area of the small circle, $A_L$ represent the area of the large circle, and let $r$ represent the radius of the large circle. Then, because the small circle has a radius of one unit, $A_s = \pi$. Also, $A_L = \pi r^2$ and $2A_s = A_L$. Thus, $2\pi = \pi r^2$, $r = \sqrt{2}$, and $R = \sqrt{2} - 1$.

b) (8 pts) Let $A_s$ represent the area of the small triangle and let $A_L$ represent the area of the larger triangle. Let $s$ represent the length of a side of the larger equilateral triangle and let $h$ represent the height of the larger triangle. Then, because the small equilateral triangle has sides of length one unit and a height of $\frac{\sqrt{3}}{2}$ units, $A_s = \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{4}$.

In the larger triangle, $h = \frac{\sqrt{3}}{2} s$, so $A_L = \frac{1}{2} hs = \frac{1}{2} \left( \frac{\sqrt{3}}{2} s \right) s = \frac{\sqrt{3}}{4} s^2$. But,

$2A_s = A_L$, so $2 \left( \frac{\sqrt{3}}{4} s^2 \right) \Rightarrow s^2 = 2 \Rightarrow s = \sqrt{2}$ and $h = \left( \frac{\sqrt{3}}{2} \right) \sqrt{2} = \frac{\sqrt{6}}{2}$.

However, (see figure) $h = \frac{\sqrt{3}}{2} + x + D = \frac{\sqrt{3}}{2} + 2D + D = \frac{\sqrt{3}}{2} + 3D$. Setting the two expressions for $h$ equal to each other,

$\frac{\sqrt{6}}{2} = \frac{\sqrt{3}}{2} + 3D \Rightarrow 3D = \frac{\sqrt{6} - \sqrt{3}}{2} \Rightarrow D = \frac{\sqrt{6} - \sqrt{3}}{6} = \frac{\sqrt{3}(\sqrt{2} - 1)}{6}$.

$x = 2D$
c) (4 pts.) The top-most ball, and the middle ball on the right side of the 3x3 square, are, by symmetry, both tangent to the centerline (altitude) of the triangle that forms the right face of the pyramid. If we call \((0,0,0)\) the center of the top ball and \((2,0,-2\sqrt{2})\) the center of the other ball, the great circles in the plane \(y = 0\) have the centerline of the triangular face as a common tangent.

That common tangent has slope \(\frac{\Delta z}{\Delta x}\), the slope of the line joining their centers, i.e., \(\frac{-2\sqrt{2}}{2} = -\sqrt{2}\).

The radius through \(z = 0\) in the plane \(y = 0\) has slope \(\frac{\Delta z}{\Delta x} = -\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\). Thus, the point of tangency satisfies \(z = \frac{x}{\sqrt{2}}\) and \(x^2 + z^2 = 1\).

\[
\therefore x^2 + \frac{1}{2}x^2 = 1
\]

\[
\frac{3}{2}x^2 = 1
\]

\[
x = \frac{2}{\sqrt{3}}, \quad z = \frac{x}{\sqrt{2}} = \frac{1}{\sqrt{3}}
\]

The altitude (centerline) of the triangular face has equation (in plane \(y = 0\)) of \(z = -\sqrt{3} \left( x - \frac{2}{\sqrt{3}} \right) = -\sqrt{2}x + \frac{2}{\sqrt{3}}\).

\[
\therefore z = -\sqrt{2}x + \frac{3}{\sqrt{3}} = -\sqrt{2}x + \sqrt{3}
\]

Thus, the \(z\)-intercept (the peak of the pyramid) is \(\sqrt{3} m\) above the center of the highest ball. So the height for the pyramid is \(c = 9\sqrt{2} + 1 + \sqrt{3}\) meters. Now if we re-coordinatize the plane \(y = 0\), so that the origin is at the center of the square, then the centerline of the triangular face still has slope \(-\sqrt{2}\) and \(z\)-intercept \(c\). So, it has equation \(z = -\sqrt{2}x + c\). This intersects the \(x\)-axis at \(\frac{c}{\sqrt{2}}\). Since \(x\) represents the distance from the center of the square base to the middle of the right hand edge, each side of the square base has length \(2\left(\frac{c}{\sqrt{2}}\right) = \sqrt{2}c\) meters. The volume of the pyramid is therefore \(\frac{1}{3}(\sqrt{2}c)^2(c) = \frac{2}{3}c^3\) cubic meters (where \(c = 9\sqrt{2} + 1 + \sqrt{3}\)).
3. a) (1 pt.) Suppose \( f(x) = ax + b \) has a fixed point. Then \( \exists x_0 \) such that
\[
x_0 = ax_0 + b \quad \text{or} \quad 0 = (a-1)x_0 + b
\]
which has the solution \( x_0 = \frac{-b}{a-1} \) as long as \( a \neq 1 \). If \( a = 1 \) and \( b = 0 \), then every point of \( f \) is a fixed point; but if \( a = 1 \) and \( b \neq 0 \), then \( f \) has no fixed point.

b) (1 pt.) If \( a \neq 1 \), \( x_0 = \frac{-b}{a-1} \) is a fixed point.

c) If there is a fixed point of \( f_c \), then \( x_0^2 - c = x_0 \) or \( x_0^2 - x_0 - c = 0 \) which has one solution when \( (-1)^2 - 4(1)(-c) = 1 + 4c = 0 \);
has two solutions when \( 1 + 4c > 0 \);
has no solutions when \( 1 + 4c < 0 \).

\[
\therefore \quad \text{i) (2 pts.) When } 1 + 4c > 0 \text{ or } c > \frac{-1}{4}, \text{ there are two fixed points.}
\]

\[
\ii) \quad \text{(2 pts.) When } 1 + 4c < 0 \text{ or } c < \frac{-1}{4}, \text{ there are no fixed points.}
\]

\[
\iii) \quad (2 \text{ pts.}) \quad x_0 = \frac{1 \pm \sqrt{1 + 4c}}{2}
\]

\[
\therefore \quad \frac{1 + 1}{2} \sqrt{1 + 4c} \quad \text{and} \quad \frac{1 - 1}{2} \sqrt{1 + 4c} \quad \text{are the fixed points.}
\]

d) There are many examples, e.g., \( f(x) = x^3 \) and \( f(x) = k \sin x \) \((1 < k < \pi)\).

4. a) (2 pts.) It is easiest to note that the sum of the first \( n \) squares is \( \frac{n(n+1)(2n+1)}{6} \) which equals 385 when \( n = 10 \).

b) (4 pts.) Consider any ball (with center \( A \)) and the 4 supporting balls (with centers \( B_1, B_2, B_3, B_4 \)). These 5 centers form a pyramid with a square base. One side of the square base is 2 meters. The center \( A \) is 2 meters from each of the \( B_i \) since the sphere is tangent to each of the \( 4 \) balls. If we let \( C \) be the centroid of this square, then the distance from \( C \) to each \( B_i \) is half of the diagonal or \( \sqrt{2} \) meters. Now \( \forall (i = 1, 2, 3, 4) \) \( \Delta ACB_i \) is a right triangle with a hypotenuse of length \( 2m \) and one leg of length \( \sqrt{2} m \). By the Pythagorean Theorem, the other leg must also have length \( \sqrt{2} m \). Thus the centers of each "level" of balls is \( \sqrt{2} m \) above the "level" of the centers of the supporting balls. The floor is \( 1m \) below the lowest level of centers and the top of the highest ball is \( 1m \) above the highest level. There are 10 levels, so the "height" of the pyramid (from the floor to the top of the highest ball is \( 2 + 9\sqrt{2} \) meters.)
5. a) (2 pts.) There are \( C(21,3) = 1330 \) possible 3 card hands. There are \( (5)(3)(3)(3) = 135 \) possible "straights" (including straight flushes which were not mentioned). The probability is \( \frac{135}{1330} = \frac{27}{266} \) that the hand will contain a straight.

b) (2 pts.) There are 1330 possible hands and \( 3xC(7,3) = 105 \) possible flushes (including the 15 straight flushes). The probability is therefore \( \frac{105}{1330} = \frac{3}{38} \) that the hand will contain a flush.

c) (3 pts.) There are now \( C(22,3) = 1540 \) possible hands. There are 105 hands (see b) that are flushes without a joker and \( 3xC(7,2) = 63 \) hands containing a joker that are flushes. The probability is therefore \( \frac{105+63}{1540} = \frac{168}{1540} = \frac{6}{55} \) that a 3 card hand will be a flush.

d) (3 pts.) Let \( s \) equal the number of suits and \( d \) equal the number of denominations. We need to find \( s \) and \( d \) so that the number of possible straights is equal to the number of possible flushes. The number of straights without a joker is \( (d-2)(s^3) \). The number of straights with a joker must be counted by considering those of the form 1,2,joker; 2,3,joker; ..., (d-1),joker (and there are \((d-1)s^2\) of those) plus those of the form 1,3,joker; 2,4,joker; ... ; (d-2),d,joker (and there are \((d-2)s^2\) of those). So there are a total of \( (d-2)s^3 + (2d-3)s^2 \) possible straights.

There are \( s \left( \frac{d(d-1)(d-2)}{6} \right) \) possible flushes without a joker and \( s \left( \frac{d(d-1)}{2} \right) \) possible flushes with a joker. This gives a total of \( s \left( \frac{d(d-1)(d-2)}{6} \right) + s \left( \frac{d(d-1)}{2} \right) = \left( \frac{s}{6} \right) d(d-1)(d-1) + \left( \frac{s}{2} \right) d \) or \( \left( \frac{s}{6} \right) d(d^2-1) \) possible flushes.

We are assuming \( s>1 \) and \( d>3 \). We need to find an integer solution to the equation: \( (d-2)s^3 + (2d-3)s^2 = \left( \frac{s}{6} \right) d(d^2-1) \) or (factoring an \( s \) out of both sides) \( (d-2)s^3 + (2d-3)s = \left( \frac{d(d^2-1)}{6} \right) \).

Proceeding by trial and error, when \( d = 11 \), this becomes \( 9s^2 + 19s = 220 \) which has a solution \( s = 4 \). So a deck consisting of 4 suits and 11 denominations has an equal likelihood of producing a straight and a flush in a 3-card hand.