MMPC	CODE	NUMBER			

FORTIETH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 11, 1996

INSTRUCTIONS
(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page, clearly write the **problem number**. If you use additional paper for your answer, **check the appropriate box** and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. If you are unable to solve a particular problem, partial credit will be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

7. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference materials, computational aids, or any other aids is likewise prohibited. Please note that calculators are <u>not</u> allowed on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. It is not necessary to return scratch paper on which routine numerical calculations have been made.

8. You may now open the test booklet.

Problem	Score
1	
2	
3	
4	
5	
TOTAL	

1. An *Egyptian Fraction* has the form 1/n, where n is a positive integer. In ancient Egypt, these were the **only** fractions allowed. Other fractions between zero and one were always expressed as a sum of **DISTINCT** Egyptian fractions. For example, 3/5 was seen as 1/2 + 1/10, or 1/3 + 1/4 + 1/60. The preferred method of representing a fraction in Egypt used the "greedy" algorithm, which, at each stage, uses that Egyptian fraction which eats up as much as possible of what is left of the original fraction. Thus the greedy fraction for 3/5 would be 1/2 + 1/10.

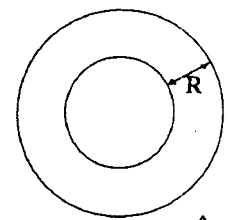
a. (2 pts.) Find the greedy Egyptian fraction representation for 2/13.
b. (2 pts.) Find the greedy Egyptian fraction representation for 9/10.
c. (2 pts.) Find the greedy Egyptian fraction representation for 2/(2k+1), where k is a positive integer.
d. (4 pts.) Find the greedy Egyptian fraction representation for the fraction 3/(6k+1), where k is a positive integer.

Check the appropriate box if your solution is continued on the

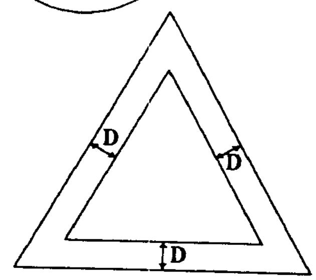
inside back cover (page 7) ☐

on inserted additional page ☐

2\. a. (2 pts.) The smaller of two concentric circles has radius one unit. The area of the larger circle is twice the area of the smaller circle. Find R, the difference in their radii.



b. (8 pts.) The smaller of two identically oriented equilateral triangles has each side one unit long. The smaller triangle is "centered" within the larger triangle so that the perpendicular distance between parallel sides is always the same number D. If the area of the larger triangle is exactly twice the area of the smaller triangle, find D.



Check the appropriate box if your solution is continued on the

inside back cover (page 7) ☐

on inserted additional page ☐

3. Suppose that the domain of a function f(x) is the set of real numbers and that f takes values in the set of real numbers.

A real number, x_0 , is a **fixed point** of f(x) if $f(x_0) = x_0$.

a. (1 pt.) Let f(x) = a x + b. For which a does f(x) have a fixed point?
b. (1 pt.) Find, in terms of a and b , the fixed point of f(x) = ax + b, if it exists.

c. Consider the functions $f_c(x) = x^2 - c$.

i. (2 pts.) For which values of c are there two different fixed points?
ii. (2 pts.) For which values of c are there no fixed points?
iii. (2 pts.) In terms of c, find the value(s) of the fixed point(s).

d. (2 pts.) Find an example of a function that has exactly 3 fixed points.

Check the appropriate box if your solution is continued on the

inside back cover (page 7) ☐

on inserted additional page ☐

4. A square based pyramid is made out of rubber balls. There are 100 balls on the bottom level, 81 on the next level, etc., up to 1 on the top level.

a. (2 pts.) How many balls are there in the pyramid?
b. (4 pts.) If each ball has a radius of 1 meter, how tall is the pyramid?
c. (4 pts.) What is the volume of the solid that you create if you place a plane against each of the four sides and the base of the balls?

Check the appropriate box if your solution is continued on the

inside back cover (page 7) ☐

on inserted additional page ☐

5. We wish to consider a "general" deck of cards which is specified by a number of suits, a sequence of denominations and a number (possibly 0) of Jokers. The deck will consist of exactly one card of each denomination from each suit, plus the Jokers which are "wild" and can be counted as any possible card of any suit. For example, a standard deck of cards consists of 4 suits and 13 denominations (and 0 Jokers)

a. (2 pts.) For a deck with 3 suits and seven denominations $\{1,2,3,4,5,6,7\}$ and no Jokers, find the probability that a 3 card hand will be a "Straight". A Straight consists of 3 cards in sequence (e.g. 1♥ , 3♣ , 2♦ , but not 6♥, 7♠, 1♦)

b. (2 pts.) For a deck with 3 suits and seven denominations and no Jokers, find the probability that a 3 card hand will consist of 3 cards of the same suit (i.e. a "Flush").

c. (3 pts.) Consider a deck with 3 suits and 7 denominations and one Joker. Compute the probability that a 3 card hand will be a straight and also the probability that a 3-card hand will be a flush if dealt at random from such a deck.

d. (3 pts.) Find a number of suits and the length of the denomination sequence that would be required if a deck is to contain one joker and is to have identical probabilities for a "straight" and a "flush", when a 3-card hand is dealt. The answer that you find must be an answer such that a "flush" and a "straight" are possible but not always certain to occur.

Check the appropriate box if your solution is continued on the

inside back cover (page 7) ☐

on inserted additional page ☐

This page may be used to continue your solution. Identify the problem number.

Problem

Check the box if your solution is continued on additional paper