40TH Annual Michigan Mathematics Prize Competition
PART I Answer Key

1. a $f(g(x\text{-}1))=f[(x-1)^2+1]=\sqrt{\left[(x-1)^2+1\right]-1}=\sqrt{(x-1)^2}=|x-1|$
2. e
3. b $p=s+d \Rightarrow 3p=3s+3d$
   $2p=3d \qquad -2p= \quad -3d$
   $p=3s$
4. e The last two digits of 1996^{1996} can be found by finding $1996^{1996}\ mod\ 100$. Since
   $1996 \equiv -4 (mod\ 100)$, $1996^{1996} \equiv (-4)^{1996} (mod\ 100) \equiv \left((-4)^4\right)^{499} (mod\ 100) \equiv (56)^{499} (mod\ 100)$
   $\equiv (56)^{400}(56)^{99} (mod\ 100) \equiv \left((56)^4\right)^{100}\left((56)^3\right)^{33} (mod\ 100) \equiv (96)^{100}(16)^{33} (mod\ 100)$
   $\equiv \left((96)^4\right)^{25}\left((16)^3\right)^{11} (mod\ 100) \equiv (56)^{25}(96)^{11} (mod\ 100) \equiv \left((56)^5\right)^5 (96)^6 (96)^5 (mod\ 100)$
   $\equiv (76)^5(96)(76) (mod\ 100) \equiv (76)^6(96) (mod\ 100) \equiv (76)(96)\ (mod\ 100) \equiv 72\underline{96} (mod\ 100)$
5. b $lcm(21,96)=672=\underline{7}\cdot 96$
6. b There are 6 out of 36 ways to roll a seven with one pair of dice, so the answer is
   $\frac{1}{6}(1728)=288$.
7. d Let d be the distance from your house to your friend's house in miles and let t be the time in hours for the round trip. Then $\frac{d}{10}+\frac{d}{8}=t$ and the average velocity for the round trip is
   $\frac{2d}{t}=\frac{2d}{\frac{d}{10}+\frac{d}{8}}=\frac{2d}{\frac{8d+10d}{80}}=\frac{160d}{18d}\approx 8.89 mph.$
8. a $a^2+(a+3)^2=(a+1)^2+(a+2)^2 \Rightarrow a^2+a^2+6a+9=a^2+2a+1+a^2+4a+4$
   $\Rightarrow 2a^2+6a+9=2a^2+6a+5 \Rightarrow 9=5$, so there are no solutions.
9. c

Week	#adult females	#infant females	#adult males	#infant males	Total
0	50	0	50	0	100
3	50	150	50	150	400
6	200	150	200	150	700
9	350	600	350	600	1900
12	950	1050	950	1050	4000

10. e
11. d The number of ways to seat 6 people at a round table is $\frac{6!}{6}=5!=120$. We divide by 2, obtaining the correct answer of 60, since the twins are indistinguishable.
12. a $65\frac{0.3}{0.5}=39mph$
13. a $[(3*2)\#5]\#1=[3^2+2(5)]+2(1)=21$
14. c

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15. e $\log_{32} 256 = \log_{2^5} 2^8 = \frac{8}{5}$ since $(2^5)^{\frac{8}{5}} = 2^8$. Similarly, $\log_{16} 32 = \log_{2^4} 2^5 = \frac{5}{4}$. So $\log_{32} 256 - \log_{16} 32 = \frac{8}{5} - \frac{5}{4} = \frac{7}{20} = 0.35$.

16. c Using the approximation of πx^2 for the area, we get $\pi x^2 = \pi(4000 \sin 23.5°)^2 \approx 8x10^6\ m^2$.

17. a There are 5(6) = 30 different outcomes from the dreidel (first coordinate) and die(second coordinate). Only (1,6), (2,5), (3,4), (4,5) & (5,6) result in a sum of 7, so the probability is $\frac{5}{30} = \frac{1}{6}$.

18. b $\log \pi \approx 0.4971498727$, so $\pi \approx 10^{0.4971498727}$ and $\pi^{3450} \approx \left(10^{0.4971498727}\right)^{3450} \approx 10^{1715}$.

19. a Let b be the base. Then $(b+8)(b+7)=b^2+9b+1 \Rightarrow b^2+15b+56 = b^2+9b+1 \Rightarrow b^2-6b-55=0 \Rightarrow b = 11$ (or b =-5).

20. e By the Pythgorean theorem, $x^2 + (x+d)^2 = (x+2d)^2 \Rightarrow 2x^2+2dx+d^2= x^2+4dx +4d^2 \Rightarrow x^2-2dx-3d^2=0 \Rightarrow (x-3d)(x+d)=0 \Rightarrow x=3d, x+d=4d,$ and $x+2d=5d.$
Since only 95 is a multiple of 3, 4, or 5, it must be the correct answer.

21. d $4^x - 4^{x-1} = 24 \Rightarrow 4^{x-1}(4-1) = 4^{x-1}(3) = 24 \Rightarrow 4^{x-1} = 8 \Rightarrow 2^{2(x-1)} = 2^3 \Rightarrow 2(x-1) = 3 \Rightarrow x = \frac{5}{2}$. Then $8^x - 8^{x-1} = 8^{\frac{5}{2}} - 8^{\frac{3}{2}} = \left(2^3\right)^{\frac{5}{2}} - \left(2^3\right)^{\frac{3}{2}} = 2^{\frac{15}{2}} - 2^{\frac{9}{2}} = 128\sqrt{2} - 16\sqrt{2} = 112\sqrt{2}$.

22. d Let x be the length of one side of the square. Then $3x = 24cm$, so $x = 8cm$ and the area of $ABCD$ is $64cm^2$.

23. c

#pennies	#thrup'pennies	#ha'pennies
3		4
2	1	
2		6
1	1	2
0	1	4

for a total of 5 ways.

24. c $2m + e = \$5.72$ and $2m + 2e + b = \$8.00$. Subtracting the first equation from the second gives $e + b = \$2.28$ so her change will be \$.72.

25. a Since 12"=**6**(2"), 24"=**12**(2"), and 5"=**2**(2") + 1", the answer is 6(12)(2) = 144.

26. d Since the two leftmost digits tell us $1\#_{20} + \#_{20} = 20_{20}$, in base 10, $20 + 2\# = 40$ and $\# = 10$.

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27. d

#in T1	#in T2	#of ways
1	9	$\binom{10}{1} = 10$
2	8	$\binom{10}{2} = 45$
3	7	$\binom{10}{3} = 120$
4	6	$\binom{10}{4} = 210$
5	5	$\dfrac{\binom{10}{5}}{2} = \dfrac{252}{2} = 126$

Divide by two since the two teams in this case are indistinguishable. The total is then 511.

28. c Because of the 5-cycles, at least 3 colors are needed.

29. c There are 366 days in 1996 (a leap year) and the clock loses 5 *min/day* for a total of 1830*min* = 30.5*hours*. This is equivalent to losing 6.5 *hours*, so the clock reads 5:30pm.

30. a $x + \sqrt{2}x = 1 \Rightarrow (1+\sqrt{2})x = 1 \Rightarrow x = \dfrac{1}{1+\sqrt{2}} = \sqrt{2} - 1$

31. d Let x be the number of people in the group originally. Then $\dfrac{30.8x + 4(32)}{x+4} = 31$
$\Rightarrow 30.8x + 128 = 31x = 124 \Rightarrow 0.2x = 4 \Rightarrow x = 20$, so there are now 24 people.

32. c Let a and b be the two numbers. If a= 0, then the probability that the distance between a and b is > 5 is 0.5 since any b >5 will make the distance greater than 5. Similarly, if a = 10, the desired probability is 0.5. If a=5, then the probability is 0 since no b in [0,10] will be more that 5 away. If a=2.5, then any b in [0,10] greater than 7.5 will make that distance between a and b greater than 5; thus this probability will be 0.25. Thus, the graph of the probability distribution *P(t)* looks like the figure below. So, the answer is given by $\dfrac{1}{10}\int_0^{10} P(t)dt = \dfrac{1}{10}\left(\dfrac{5}{2}\right) = \dfrac{1}{4}$.

33. b Since $\cos^{-1}(-0.25) \approx 104°$, the angle must be opposite the largest side. So, using the Law of Cosines, $(k+10)^2 = k^2 + (k+5)^2 - 2k(k+5)(-0.25) \Rightarrow 1.5k^2 - 7.5k - 75 = 0 \Rightarrow k = 10$ or $k = -5$. But k cannot be -5 so k = 10.

34. b $2x - 3y = 5 \Rightarrow y = \frac{2}{3}x - \frac{5}{3}$ Therefore, since x and y must both be positive integers, $x \equiv 1 \pmod 3$. Then, x = 4, 7, 10, ..., 97 give positive y values less than 99 so there are 32 solutions.

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35. e $(2{,}000{,}000gal)(231in^3/gal)\left(\frac{1ft^3}{12^3in^3}\right)\left(0.3048^3\frac{m^3}{ft^3}\right)$=$7570m^3$ water. To get the depth, take this volume and divide by the area of one acre in m^2. One $acre = (43560ft^2)\left(0.3048^2\frac{m^2}{ft^2}\right) \approx$ $4047m^2$. Thus the answer is $\frac{7570m^3}{4047m^2} \approx 1.87m.$

36. a Let r be the average speed on the superhighway (in mph). Then the time in hours for the scenic route is $\frac{231}{r-18} = \frac{150}{r} + 3$. Solving for r we obtain r=60mph so the time for the direct route is $150m/60mph = 2.5hours$.

37. b $30000(1+.98+.98^2+.98^3+...+.98^{19}) = 30000(1+.98+.98^2+.98^3+...+.98^{19})\left(\frac{1-.98}{1-.98}\right) =$ $\frac{30000(1-.98^{20})}{(1-.98)} \approx 498588$

38. d $2sin^2 3\theta = 1 \Rightarrow sin^2 3\theta = \frac{1}{2} \Rightarrow sin 3\theta = \pm\frac{\sqrt{2}}{2}$ Thus there are 4 solutions for 3θ between 0 and 360 degrees and 3(4)= 12 solutions for θ in that same range.

39. d He sees buses at 1:00, 1:05, 1:10, 1:15, and 1:20, so he sees the buses every 5 *minutes*. However, by the time he sees the bus at 1:05, the bus he saw 5 *min.* ago at 1:00 has been traveling in the opposite direction for 5 *min.* Therefore that bus is 10 *min.* ahead.

40. c Let x represent the height of the trapezoid; then $x = \cos 45° = \frac{\sqrt{2}}{2}$. The trapezoid has a base of length 2, the other base of length 2-2x, and height x so its area is given by $\frac{1}{2}x(2+(2-2x)) = x(2-x) = \frac{\sqrt{2}}{2}\left(2-\frac{\sqrt{2}}{2}\right) = \sqrt{2}-\frac{1}{2}$. The unshaded region in the trapezoid is given by $\frac{\pi}{4}$. Therefore, the percent of area **shaded** is $\frac{\sqrt{2}-\frac{1}{2}-\frac{\pi}{4}}{\sqrt{2}-\frac{1}{2}} \approx 14\%$.