Thirty-Ninth Michigan Mathematics Prize Competition
Part II Solutions

1. (a) [4 points] If each of Brian's friends needs x hours to complete the job alone, then $1/2+6/x=4/3$, so x=36/5 and 1/2 + 3/x =12/11 hours.

(b) [6 points] $\frac{1}{N}+\frac{1}{kN}+\frac{1}{k^2N}+\ldots=\frac{1}{N}\sum_{n\geq 0}\frac{1}{k^n}=\frac{1}{N}\left(\frac{1}{1-\frac{1}{k}}\right)=\frac{k}{N(k-1)}$ hours.

2. (a) [2 points] The area of the each of the portions of the circles contained inside the square is $\frac{\pi}{4}$. If A is the area of the intersection, then $\frac{\pi}{4}+\frac{\pi}{4}=\text{area(square)}+A$. So $A=\frac{\pi}{2}-1$.

(b) [3 points] If P and Q are the centers of the circles and R is the point inside the square where the circles meet, then ΔPQR is equilateral. If B is the area of the intersection, then

$B=2\times\text{area(sector of }60^\circ)-\text{area}(\Delta PQR)=2\times\pi\times\frac{1}{6}-\frac{1}{2}\times\frac{\sqrt{3}}{2}=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$.

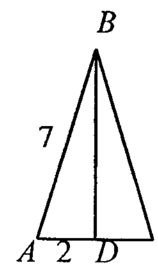
(c) [5 points] The four circles divide the square into 9 parts of three different areas. If x is the area of the intersection, y is the area of each of the four parts adjacent to the sides, and z is the area of each of the remaining four parts, then $x+4y+4z=1$, $x+2z=A$, and $x+2z+y=B$, where A and B are as in (a) and (b). The solution to this system gives $x=\frac{\pi}{3}-\sqrt{3}+1$.

3. (a) [2 points] 2

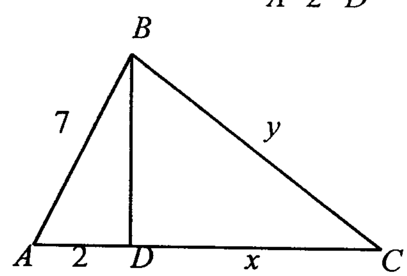
(b) [3 points] 4

(c) [5 points] Since $\left[\frac{N}{k}\right]=\left[\frac{N-1}{k}\right]+1$ if and only if k divides N, the terms of the form $\left[\frac{1996}{k}\right]$ will equal the corresponding terms $\left[\frac{1995}{k}\right]$ if and only if k doesn't divide 1996. The only terms remaining in f(1996)-f(1995) are those of the form $\left[\frac{1996}{k}\right]-\left[\frac{1995}{k}\right]$ where k divides 1996. Each of these terms will be 1. The divisors of 1996 are 1, 2, 4, 499, 998, and 1996. So f(1996)-f(1995)=6.

4. (a) [2 points] Construct a right triangle ABD with hypotenuse $AB = 7$ and $AD = 2$ (so $\cos(A) = \frac{2}{7}$) as at right. Reflect around line BD to get a good triangle with sides of length 4, 7, and 7.



(b) [3 points], (c) [5 points]
Let ABC be a good triangle as at right. Construct an altitude from B to the point D on line AC. Then triangle ABD is as in (a). Let the length of CD be x and then length of BC be y. Since the length of BD is $\sqrt{45}$, we see that $y^2 - x^2 = (y-x)(y+x) = 45$. Since x and y are integers, $y>x$, we find that $y-x=1$, $y+x=45$ or $y-x=3$, $y+x=15$ or $y-x=5$, $y+x=9$. The other good triangles have sides of length 7, 23, 24 or 7, 8, 9.



5. (a) [4 points] Let K be the number of white balls. The probability that the two balls are of the same color is $\left(\frac{K}{9}\right)\left(\frac{K-1}{8}\right)+\left(\frac{9-K}{9}\right)\left(\frac{8-K}{8}\right)$. The probability that they are of different colors is $\left(\frac{K}{9}\right)\left(\frac{9-K}{8}\right)+\left(\frac{9-K}{9}\right)\left(\frac{K}{8}\right)$. So we have $\left(\frac{K}{9}\right)\left(\frac{K-1}{8}\right)+\left(\frac{9-K}{9}\right)\left(\frac{8-K}{8}\right)=\left(\frac{K}{9}\right)\left(\frac{9-K}{8}\right)+\left(\frac{9-K}{9}\right)\left(\frac{K}{8}\right)$. Solving for K we see that K can be 6 or 3.

(b) [6 points] In the same manner as above, we obtain the following:
$\left(\frac{K}{N}\right)\left(\frac{K-1}{N-1}\right)+\left(\frac{N-K}{N-1}\right)\left(\frac{N-1-K}{N-1}\right)=\left(\frac{K}{N}\right)\left(\frac{N-K}{N-1}\right)+\left(\frac{N-K}{N}\right)\left(\frac{K}{N-1}\right)$ which reduces to $4K^2 - 4NK + N(N-1) = 0$. The roots of this equation are $K = \frac{N+\sqrt{N}}{2}$ and $K = \frac{N-\sqrt{N}}{2}$, so N must be a perfect square. Since $180<N<220$, $N=196$ is the only possibility. Then $K=105$ and $N-K=91$ or $K=91$ and $N-K=105$.