Thirty-Ninth Michigan Mathematics Prize Competition
Part II Solutions

1. (a) [4 points] If each of Brian's friends needs $x$ hours to complete the job alone, then $1/2 + 6/x = 4/3$, so $x = 36/5$ and $1/2 + 3/x = 12/11$ hours.

(b) [6 points] $\frac{1}{N} + \frac{1}{kN} + \frac{1}{k^2N} + \ldots = \frac{1}{N} \sum_{n \geq 0} \frac{1}{k^n} = \frac{1}{N} \left( \frac{1}{1 - \frac{1}{k}} \right) = \frac{k}{N(k-1)}$ hours.

2. (a) [2 points] The area of the each of the portion of the circles contained inside the square is $\frac{\pi}{4}$. If $A$ is the area of the intersection, then $\frac{\pi}{4} + \frac{\pi}{4} = \text{area(square)} + A$. So $A = \frac{\pi}{2} - 1$.

(b) [3 points] If $P$ and $Q$ are the centers of the circles and $R$ is the point inside the square where the circles meet, then $\Delta PQR$ is equilateral. If $B$ is the area of the intersection, then

$$B = 2 \times \text{area(sector of 60°)} - \text{area}(\Delta PQR) = 2 \times \pi \times \frac{1}{6} - \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

(c) [5 points] The four circles divide the square into 9 parts of three different areas. If $x$ is the area of the intersection, $y$ is the area of each of the four parts adjacent to the sides, and $z$ is the area of each of the remaining four parts, then $x + 4y + 4z = 1$, $x + 2z = A$, and $x + 2z + y = B$, where $A$ and $B$ are as in (a) and (b). The solution to this system gives $x = \frac{\pi}{3} - \sqrt{3} + 1$.

3. (a) [2 points] 2

(b) [3 points] 4

(c) [5 points] Since $\left[ \frac{N}{k} \right] = \left[ \frac{N-1}{k} \right] + 1$ if and only if $k$ divides $N$, the terms of the form $\left[ \frac{1996}{k} \right]$ will equal the corresponding terms $\left[ \frac{1995}{k} \right]$ if and only if $k$ doesn't divide 1996. The only terms remaining in $f(1996) - f(1995)$ are those of the form $\left[ \frac{1996}{k} \right] - \left[ \frac{1995}{k} \right]$ where $k$ divides 1996. Each of these terms will be 1. The divisors of 1996 are 1, 2, 4, 499, 998, and 1996. So $f(1996) - f(1995) = 6$. 
4. (a) [2 points] Construct a right triangle $ABD$ with hypotenuse $AB = 7$ and $AD = 2$ (so $\cos(A) = \frac{2}{7}$) as at right. Reflect around line $BD$ to get a good triangle with sides of length 4, 7, and 7.

(b) [3 points], (c) [5 points]
Let $ABC$ be a good triangle as at right. Construct an altitude from $B$ to the point $D$ on line $AC$. Then triangle $ABD$ is as in (a). Let the length of $CD$ be $x$ and then length of $BC$ be $y$. Since the length of $BD$ is $\sqrt{45}$, we see that

$$y^2 - x^2 = (y-x)(y+x) = 45.$$ Since $x$ and $y$ are integers, $y>x$, we find that $y-x = 1$, $y+x = 45$ or $y-x = 3$, $y+x = 15$ or $y-x = 5$, $y+x = 9$. The other good triangles have sides of length 7, 23, 24 or 7, 8, 9.

5. (a) [4 points] Let $K$ be the number of white balls. The probability that the two balls are of the same color is $\frac{K}{9} \cdot \frac{K-1}{8} + \frac{9-K}{9} \cdot \frac{8-K}{8}$. The probability that they are of different colors is $\frac{K}{9} \cdot \frac{9-K}{8} + \frac{9-K}{9} \cdot \frac{K}{8}$. So we have

$$\frac{K}{9} \cdot \frac{K-1}{8} + \frac{9-K}{9} \cdot \frac{8-K}{8} = \frac{K}{9} \cdot \frac{9-K}{8} + \frac{9-K}{9} \cdot \frac{K}{8}.$$ Solving for $K$ we see that $K$ can be 6 or 3.

(b) [6 points] In the same manner as above, we obtain the following:

$$\frac{K}{N} \cdot \frac{K-1}{N-1} + \frac{N-K}{N-1} \cdot \frac{N-1-K}{N-1} = \left(\frac{K}{N} \cdot \frac{N-K}{N-1}\right) + \left(\frac{N-K}{N} \cdot \frac{K}{N-1}\right)$$

which reduces to $4K^2 - 4NK + N(N-1) = 0$. The roots of this equation are $K = \frac{N + \sqrt{N}}{2}$ and $K = \frac{N - \sqrt{N}}{2}$, so $N$ must be a perfect square. Since $180 < N < 220$, $N = 196$ is the only possibility. Then $K = 105$ and $N-K = 91$ or $K = 91$ and $N-K = 105$. 
