MMPC	CODE	NUMBER			

THIRTY-NINTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 6, 1995

INSTRUCTIONS
(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page, clearly write the **problem number**. If you use additional paper for your answer, **check the appropriate box** and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. If you are unable to solve a particular problem, partial credit will be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

7. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference materials, computational aids, or any other aids is likewise prohibited. Please note that calculators are <u>not</u> allowed on this exam. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. It is not necessary to return scratch paper on which routine numerical calculations have been made.

8. You may now open the test booklet.

Problem	Score
1	
2	
3	
4	
5	
TOTAL	

1.

[4 points] (a) Brian has a big job to do that will take him two hours to complete. He has six friends who can help him. They all work at the same rate, somewhat slower than Brian. All seven working together can finish the job in 45 minutes. How long will it take to do the job if Brian worked with only three of his friends?

[6 points] (b) Brian could do his next job in N hours, working alone. This time he has an unlimited list of friends who can help him, but as he moves down the list, each friend works more slowly than those above on the list. The first friend would take kN $(k > 1)$ hours to do the job alone, the second friend would take k^2N hours alone, the third friend would take k^3N hours alone, etc. Theoretically, if Brian could get all his infinite number of friends to help him, how long would it take to complete the job?

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

on inserted additional paper ☐

2.

[2 points] (a) The centers of two circles of radius 1 are two opposite vertices of a square of side 1. Find the area of the intersection of the two circles.

[3 points] (b) The centers of two circles of radius 1 are two consecutive vertices of a square of side 1. Find the area of the intersection of the two circles and the square.

[5 points] (c) The centers of four circles of radius 1 are the vertices of a square of side 1. Find the area of the intersection of the four circles.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

on inserted additional paper ☐

3. For any real number x, $[x]$ denotes the greatest integer that does not exceed x. For example, $[7.3] = 7$, $[10/3] = 3$, $[5] = 5$. Given natural number N, denote as $f(N)$ the following sum of N integers:

$$f(N) = [N/1] + [N/2] + [N/3] + \cdots + [N/N].$$

[2 points] (a) Evaluate $f(7) - f(6)$.

[3 points] (b) Evaluate $f(35) - f(34)$.

[5 points] (c) Evaluate (with explanation) $f(1996) - f(1995)$.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

on inserted additional paper ☐

4. We will say that triangle ABC is good if it satisfies the following conditions: $AB = 7$, the other two sides are integers, and $\cos A = \frac{2}{7}$.

[2 points] (a) Find the sides of a good isosceles triangle.

[3 points] (b) Find the sides of a good scalene (i.e. non-isosceles) triangle.

[5 points] (c) Find the sides of a good scalene triangle other than the one you found in (b) and prove that any good triangle is congruent to one of the three triangles you have found.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

on inserted additional paper ☐

5.

[4 points] (a) A bag contains nine balls, some of which are white, the others are black. Two balls are drawn at random from the bag, without replacement. It is found that the probability that the two balls are of the same color is the same as the probability that they are of different colors.

How many of the nine balls were of one color and how many of the other color?

[6 points] (b) A bag contains N balls, some of which are white, the others are black. Two balls are drawn at random from the bag, without replacement. It is found that the probability that the two balls are of the same color is the same as the probability that they are of different colors. It is also found that $180 < N < 220$.

Find the exact value of N and determine how many of the N balls were of one color and how many of the other color.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

on inserted additional paper ☐

This page may be used to continue your solution. Identify the problem number.

Problem ☐

Check the box if your solution is continued on inserted additional paper ☐

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America

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