Thirty-Ninth Michigan Mathematics Prize Competition
Part I

The answer key to Part One of the Thirty-Ninth annual Michigan Math Prize Competition, which was given on October 11, 1995 is as follows:

1. D	11. B	21. A	31. B
2. A	12. E	22. A	32. C
3. A	13. C	23. B	33. D
4. E	14. D	24. C	34. A
5. D	15. C	25. C	35. C
6. C	16. D	26. C	36. D
7. D	17. B	27. D	37. B
8. A	18. C	28. B	38. D
9. D	19. D	29. C	39. E
10. B	20. A	30. B	40. B

Detailed Solutions to selected problems

2. $\sin^{-1}(0.345) \approx 0.352$, which is not between $\frac{\pi}{2}$ and π. To get a number with the same sine, we subtract from π: $\pi - 0.352 \approx 2.789$.
3. Let R be the radius of the outer circle and r the radius of the inner circle. The difference in area is 5 times the area of the smaller circle. So we have $\pi R^2 - \pi r^2 = 5\pi r^2$. We can rewrite this equation to see that $\frac{R^2}{r^2} = 6$ or that $\frac{R}{r} = \sqrt{6}$.
4. In the expression $\frac{x}{2} + \frac{2}{x} + \cdots + \frac{x}{2} + \frac{2}{x}$ containing 72 terms, there are 36 terms of the form $\frac{x}{2}$ and 36 of the form $\frac{2}{x}$. Note that $36\left(\frac{x}{2}\right) + 72\left(\frac{2}{x}\right) = \frac{18x^2 + 72}{x}$.
7. Let w_{30} and h_{30} be the width and height of a 30 inch screen and w_{20} and h_{20} the width and height of a 20 inch screen. We know that $\frac{w_{30}}{h_{30}} = \frac{w_{20}}{h_{20}}$. Call this common ratio r. So $w_{30} = rh_{30}$ and $w_{20} = rh_{20}$. The ration of the areas of the screens is $\frac{w_{30}h_{30}}{w_{20}h_{20}} = \frac{rh_{30}^2}{rh_{20}^2} = \frac{h_{30}^2}{h_{20}^2}$. A

30 inch screen has a 30 inch diagonal, so $\left(r^2+1\right)h_{30}^2 = 30^2$ or $h_{30}^2 = \dfrac{30^2}{\left(r^2+1\right)}$. Similarly,

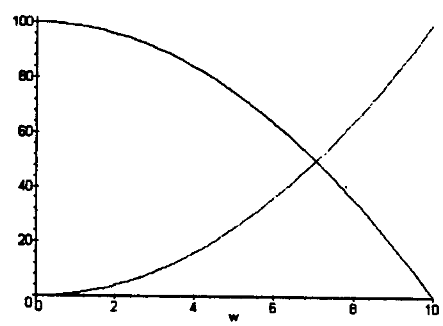
$h_{20}^2 = \dfrac{20^2}{\left(r^2+1\right)}$. So the ratio of areas is $\dfrac{h_{30}^2}{h_{20}^2} = \dfrac{\dfrac{30^2}{\left(r^2+1\right)}}{\dfrac{20^2}{\left(r^2+1\right)}} = \dfrac{9}{4}$.

8. The left hand side of the equation simplifies to $\dfrac{17}{30} \approx .567$ which is bigger than 0.5. Since the variables are all positive integers, if $x>1$, then the right hand side will be less than 0.5. So we must have $x=1$.

10. $16\sin(x) = 32\cos(2x) - 33 = 32\left(1-2\sin^2(x)\right) - 33$. This leads to the equation

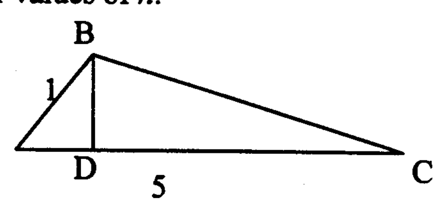
$64\sin^2(x) + 16\sin(x) + 1 = 0$. Factoring we see that $\sin(x) = -\dfrac{1}{8}$.

14. Let l be the length and w the width of the rectangle. Then $l^2 + w^2 = 100$. So the area of the rectangle is $w\sqrt{100-w^2} = \sqrt{w^2\left(100-w^2\right)}$. We assume that $0<w<10$. The graphs of $f(w) = w^2$ and $g(w) = 100 - w^2$ are both parabolas, shown at right. Note that if $f(w) = k$ then $g(w) = 100 - k$, $0 < k < 100$. So the area of the rectangle is $\sqrt{f(w)g(w)} = \sqrt{k(100-k)}$. Now $k(100-k)$ is a parabola with maximum value when $k=50$. So the maximum area is $\sqrt{(50)(50)} = 50$.



15. Each square has sides of length 8. If we are at a point to the right of P where there are equal areas above and below PR, then there are an equal number of rectangles, say n of them, above and below PR. The point on the last rectangle farthest from P on PR is $2n \times 8 + (2n-1) \times 4 = 24n - 4$ units from P. At each of the points in the interval $[24n-4, 24n]$ there will be equal areas above and below PR. Now 70 is the only point given that lies in an interval of the form $[24n-4, 24n]$ for integer values of n.

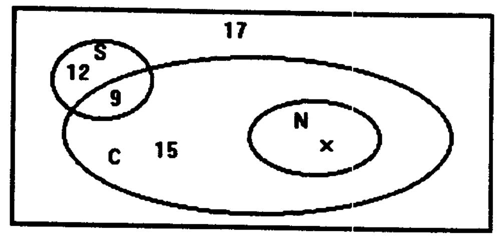
17. Call the angles in the triangle A, B, and B and let $\alpha, \alpha+r, \alpha+2r$ be their measures. The sum of these angles is 180°, so the measure of angle A is $\alpha + r = 60^\circ$. The height of the triangle, BD, has length $\sin(A) = \sin(60^\circ) = \dfrac{\sqrt{3}}{2}$. So the area is given by b).



19. The sum of the integers from 1 to n is $\frac{n(n+1)}{2}$. So the sum of the first 1995 integers is S=1991010. The sum of the integers from 1996 to N is the sum of the integers from 1 to N minus the sum of the integers from 1 to 1995 or $\frac{N(N+1)}{2}-S$. We want N so that $\frac{N(N+1)}{2}-S \approx S$. If we solve $\frac{N(N+1)}{2}-S=S$ for real N we get N to be approximately 2821.563120. The closest integer N is 2822.

20. Using the law of sines we can get c). The law of cosines will give us b). Since $|AB|>|BC|$, then measure of angle A is less than the measure of angle C. This forces e) and d) to be true. By the process of elimination, a) must be the answer.

21. Let S be the set of silly items, C the set of crazy items, and N the set of nutty items. Let x be the number of nutty items. We fill in the Venn diagram at right. The total number of items is 60 so x must be 7.



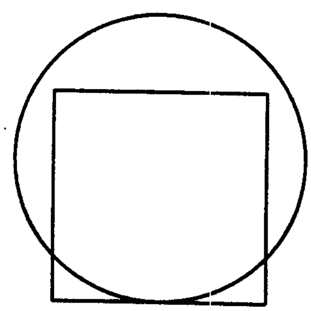
22. The law of cosines gives $\cos(A)=d$, so $\cos(A)<0.05$. Computing the inverse cosine we see that A is between 87 and 93 degrees.

23. An illuminated point is one of the form $\left(a^2,b^2\right)$ for integers a and b. If such a point lies inside the circle $x^2+y^2=27^2$, then $a^4+b^4 \le 27^2$. So the largest either a and b can be is 5. Now just consider all the possibilities.

24. Using the recurrence relation and brute force we can construct the chart

n	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	5	11	19	29	41	55	71	89	109	131

28. It is not hard to see that 0, 1, 2, 4, and 8 are possible. To see that 3 points of intersection are possible, see the picture at right. The only option is answer b).



29. Let N be the number defined by $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$ (an infinite number of the $\sqrt{}$ symbols) if such a number exists. Then the sum with 1000 of the $\sqrt{}$ symbols should be close to N. Now $N=\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$, so squaring both sides and subtracting 1 gives us $N^2-1=\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}=N$. Solving this last equation for N gives us $N=\frac{1+\sqrt{5}}{2}\approx 1.6$.

30. If RS is tangent to circle O at point T, then OT is perpendicular to RS. Let O be the center of the circle of which ACB is a quarter arc. Then angles OBF and OAF are right angles. Also, since ACB is a quarter arc, angle AOB is a right angle. So quarilateral $OAFB$ must be a square. Since C is the midpoint of arc ACB, OF must pass through C. The lengths of the sides of the square $OAFB$ are 1, so the measure of OF is $\sqrt{2}$. Since the measure of OC is 1, the measure of FC is $-\sqrt{2}$. Since DE is parallel to AB, angles FDE and FED are congruent. So each has measure $45°$. It follows that DC has length $-\sqrt{2}$. So the area of triangle DEF will be $(1-\sqrt{2})^2$.

31. Since $44 = 2 \times 2 \times 11$, for $44N$ to be a perfect cube, N must have 11^2 and 4 as factors. The smallest such N is 242.

33. Let b, o, and a be the unit cost, in dollars, of bananas, oranges, and apples respectively. Assuming that the stooges all pay the same amount for each item, the information provided yields two equations:

$$b + 2o + 2a = 2$$
$$3b + o + a = 1.5$$

Since the coefficients of o and a are the same in each equation and the coefficients of b are not proportional, no information involving the same numbe of apples and oranges will give us any new information. For example, adding the two equations shows that the information in c) is not new. So d) is the answer.

35. Let A be the amount Jill earns on June 1. The total amount of money she will have earned after the ith day will be

$$A + (A+1) + (A+2) + \ldots + (A+i) = (i+1)A + (1+2+\ldots+i) = (i+1)A + \frac{i(i+1)}{2}.$$

August 31 is 29 (June days) + 31 (July days) + 31 (August days) after June 1. So Jill has earned $92A + 4186$ dollars. So $A = 27$. July 4 is 29+4 days after June 1, so Jill earns 27+33=60 dollars.

36. If $\log_{10}(x)$ is between integers a and $a+1$, then x is between 10^a and 10^{a+1}. So x will have $a+1$ digits. Since $\log_{10}(1995^{1995}) = 1995\log_{10}(1995) \approx 6583.4$, 1995^{1995} has 6584 digits.

37. Note that in all answers but one, 3 is given as a root. Let $p(x) = x^3 + 6ax^2 + ax + 30$. If 3 is a root, then $p(3) = 0$. This can happen only if $a = -1$. In this case we also have $p(5) = p(-2) = 0$.

38. $A = 44{,}000$, $i = 0.0925$, and $P = 452.85$. Solving for n yields $n \approx 180$ months or 15 years.

40. The triangle JIA is an isosceles right triangle with congruent sides of length 4. So angle JAI is a $45°$ angle. Therefore, the diagonal of square ABCD is perpendicular to line AJ. Similarly, the diagonal of square EFGH is perpendicular to line FH. The perpendicular distance from FH to AJ will be the length of AC plus half the length of EG. AC has length $8\sqrt{2}$ and EG has length $4\sqrt{2}$.