MMPC-94, Part II
Solutions

Ken

1. Let J be Jane's speed (in MPH), Y be the distance Al jogs (in miles), and Z be the distance Jane drives to meet Al (also in miles). Then Jane leaves at $6 - (Y + Z/J)$ o'clock to meet Al, so we have

$$\frac{Y}{5} = \frac{Z}{J} + (1 - \frac{Y+Z}{J}).$$

The fact that Jane arrives back home 12 minutes earlier than usual yields

$$\frac{2(Y+Z)}{J} = \frac{2Z}{J} + \frac{1}{5}.$$

Solving, we see that Jane's speed $= J = 45$ MPH. Al jogs $Y = 4\ 1/2$ miles, and, perhaps surprisingly, Z, the distance Jane travels to pick up Al, is undetermined.

2. (a) Triangles APB and ABC share the altitude from B, and have bases $|AP| = (2/3)|AC|$, so $|APB| = (2/3)|ABC| = 4$ sq. ft. Triangles APM and APB share the altitude from P, and the base $|AM| = (1/2)|AB|$, so $|AMP| = (1/2)|APB| = 2$ sq. ft.

(b) Let M' be the altitude from M to AC, and N' the altitude from N to AC. Then triangles NPN' and MPM' are congruent, so $|MM'| = |NN'|$. Hence $|CNP| = (1/2)|NN'| \cdot |PC| = (1/2)|MM'| \cdot (1/2)|AP| = (1/2) \cdot |AMP| = 1$ sq. ft.

(c) Triangles BMC and AMC share the altitude from C, and have bases $|BM| = |AM|$, so $|BMC| = |AMC|$. Also, $|AMC| = (1/2)|AC| \cdot |MM'| = (1/2)|AC| \cdot |NN'| = |ANC|$. Finally, Triangles ANC and AND share the altitude from A, and have bases $|NC| = |ND|$, so $|ANC| = |AND|$. The quadrilateral $ABCD$ is made up of 4 triangles of equal area, BMC, AMC, ANC, and AND. In particular, triangles ABC and ACD have equal area, so the area of $ABCD$ is twice the area of ABC, 12 sq. ft.

3. (a) Clearly $0 < \tan^{-1}(1), \tan^{-1}(2), \tan^{-1}(3) < \pi/2$. Also

$$\begin{aligned}
\tan(\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)) &= \frac{1 + \tan(\tan^{-1}(2) + \tan^{-1}(3))}{1 - \tan(\tan^{-1}(3) + \tan^{-1}(3))} \\
&= \frac{1 + \frac{2+3}{1-2\cdot 3}}{1 - \frac{2+3}{1-(2+3)}} \\
&= 0
\end{aligned}$$

so $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$.

(b) We require

$$\begin{aligned}
\tan(\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{2}+k) + \tan^{-1}(\frac{1}{2}+2k)) &= \frac{\frac{1}{2} + \tan(\tan^{-1}(\frac{1}{2}+k) + \tan^{-1}(\frac{1}{2}+2k))}{1 - \frac{1}{2}\tan(\tan^{-1}(\frac{1}{2}+k) + \tan^{-1}(\frac{1}{2}+2k))} \\
&= \frac{\frac{1}{2} + \frac{(\frac{1}{2}+k)+(\frac{1}{2}+2k)}{1-(\frac{1}{2}+k)(\frac{1}{2}+2k)}}{1 - \frac{1}{2}\frac{(\frac{1}{2}+k)+(\frac{1}{2}+2k)}{1-(\frac{1}{2}+k)(\frac{1}{2}+2k)}} \\
&= 0.
\end{aligned}$$

This boils down to $\frac{1}{2} + \frac{3k+1}{-2k^2 - \frac{3}{2}k - \frac{3}{4}} = 0$, which has solutions $k = 11/4,\ -1/2$. The only admissible solution is $k = 11/4$.

4. (a) Say the integers are $a-9, a-8, \cdots, a-1, a, a+1, \cdots, a+8, a+9$. Then we want $19a \approx 1000$. The closest possible approximation occurs when $a = 53$. The desired numbers are 44,45,$\cdots$,62, with sum $19 \cdot 53 = 1007$.

(b) For the sum of odd numbers to be even, there must be an even number of terms. Say they are $a-(2n-1), a-(2n+3), \cdots, a-3, a-1, a+1, a+3, \cdots, a+(2n-3), a+(2n-1)$. (So a must be even.) The sum is $2na = 1000$. The least possible a is 2, so the largest possible n is 250. The resulting sequence $-497, -495, \cdots, 495, 497, 499, 501$ has 500 terms.

If the student thinks integers must be positive, the longest possible sequence is 31,33,35,$\cdots$,69, which has 20 terms.

5. (a) Let $a = |AX|$, $b = |BX|$, etc. By a standard theorem, $ab = cd$, so $a^2 = (a/b) \cdot ab = (c/d) \cdot cd = c^2$. Therefore $a = c$ and $b = d$, so $|AB| = a + b = c + d = |CD|$.

(b) We are given $a/b > c/d > 1$, so in particular $a > b$ and $c > d$. Using again the fact that $ab = cd$ we have $a^2 = (a/b) \cdot ab > c/d \cdot cd = c^2$, so $a > c$ and consequently $b < d$. Thus $a - b > c - d > 0$. Therefore $a^2 - 2ab + b^2 > c^2 - 2cd + d^2$, so $a^2 + 2ab + b^2 > c^2 + 2cd + d^2$, so $a + b > c + d$, that is, $|AB| > |CD|$.