THIRTY-EIGHTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 12, 1994

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. **Check to insure that your six-digit code number has been recorded correctly.** Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "**e)** none of the others". If you believe none of the first four alternatives to be correct, mark **e)** in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You many now open the test booklet and begin.

1. Evaluate

$$1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{6+\frac{1}{8}}}}$$

a) 41/3 b) 75/8 c) 49/24 d) 372/283 e) 661/457

2. In triangle ACD, point B is located on side AC so that line segments AB and BD are equal, as are segments BC and CD. If angle ACD has measure 80°, what is the measure of angle CAD?

a) 15° b) 20° c) 25° d) 30° e) 35°

3. A man travels at a rate of m miles per hour for t hours, and then changes his rate to s miles per hour for the next h hours. His average speed for the entire trip is

a) $\dfrac{mt+sh}{t+h}$ b) $\dfrac{mt+sh}{2}$ c) $\dfrac{m+s}{2}$ d) $\dfrac{mt+sh}{m+s}$ e) $\dfrac{(m/t)+(s/h)}{2}$

4. The expression $\dfrac{\dfrac{x}{x-1}-\dfrac{x}{x+1}}{\dfrac{x}{x^2-1}}$ simplifies to

a) 2 b) $2x$ c) $x+1$ d) x^2+1 e) $x-1$

5. Cross 12 digits out of the 27-digit number 123456789111213141516171819 so as to obtain the smallest possible remaining 15-digit number. The remaining number is

a) 167891111111111 b) 111111116171819 c) 111111111171819
d) 111111111111819 e) 111111111111119

6. Sphere A has diameter half again as large as the diameter of sphere B. The volume of sphere A is how many times as large as the volume of sphere B?

a) $\frac{1}{8}$ b) $\frac{1}{2}$ c) 2 d) $3\frac{3}{8}$ e) 8

7. An angle of a triangle has measure m (in degrees). If $\sin m > 1/2$, then which of the following must be true?

a) $m < 30$ b) $30 < m < 150$ c) $m > 150$ d) $m > 90$ e) $m < 90$

8. For all positive x, $\log_{100} x + \log_{1000} x =$

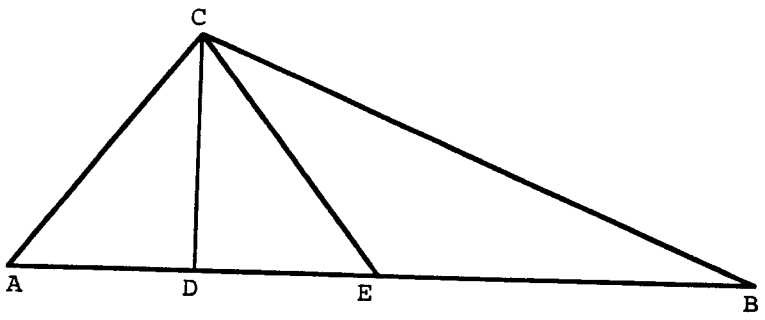
a) $\log_{1100} 2x$ b) 5 c) $5 \log_{10} \sqrt[6]{x}$ d) $\log_{10} 5x$ e) 5/6

9. Let K be any number. Then the two lines with equations $y - 3x = K$ and $6x - 2y = K$

a) coincide
b) don't coincide, but are parallel
c) are perpendicular
d) intersect, but are not perpendicular
e) More than one of these is possible.

10. Points D and E on side AB of triangle ABC are such that rays CD and CE trisect angle ACB. Also, $AC = CE = EB$. Find the ratio of the length of AD to the length of AB.

a) 1/2
b) 1/3
c) 1/4
d) 2/3
e) 1/6



11. The equation $2x + 5 = bx - 1$ has a solution x

a) for all values of b
b) for all but one value of b
c) for exactly one value of b
d) for exactly two values of b
e) for no values of b

12. A five-digit number M contains each of the digits 1,2,3,4 and 5. The number N is written with the same digits in the reverse order. (For example, if $M = 41325$, then $N = 52314$.) What is the smallest possible value of $|M - N|$?

a) -41976 b) 0 c) 1059 d) 6039 e) 5000

13. The last digit of the number $\left(\sqrt{2}\right)^{1994}$ is

a) 2 b) 4 c) 6 d) 8 e) 0

14. Square $ABCD$ has sides of length 1. Find the area common to the circles of radius 1 with centers at A and C.

a) 1/3 b) 1/2 c) $\frac{\pi}{2} - 1$ d) $\frac{\pi}{4} + 1$ e) $\frac{\pi}{4}$

15. A roofer can finish a roof in 7 hours. Each of his two assistants can work 2/3 as fast as the roofer. How long will it take the roofer and his assistants to finish the roof?

a) 2 hr., 20 min. b) 2 hr., 40 min. c) 3 hr.
d) 3 hr., 30 min. e) 3 hr., 40 min.

16. In the sequence $1, 5, 4, 1, 3, 2, \cdots$, every term, except the first and second, is equal to the absolute value of the difference between the two previous terms. Find the 1994^{th} term of this sequence.

a) -1 b) 0 c) 1 d) 1994 e) None of these.

17. The sum of the digits in the numbers from 1 to 99 (inclusive) is

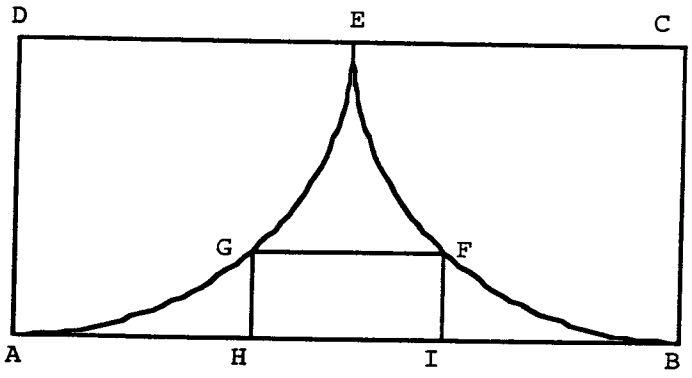
a) 45 b) 450 c) 495 d) 900 e) 4950

18. Which is the set of values of k for which the curves $x^2 + y^2 = 9$ and $y = 3 - kx^2$ intersect in exactly one point?

a) $k \leq 3$ b) $k \leq 1/6$ c) $k \leq 1/9$ d) $k = 0$ e) $k \leq 0$

19. In rectangle $ABCD$, side AB is twice as long as side BC. Arcs AE and BE are quarter circles centered at D and C, respectively. Points G and F are the midpoints of the respective circular arcs. The area of rectangle $FGHI$ comprises what percentage of the area of rectangle $ABCD$?

a) less than 8%
b) between 8% and 9%
c) between 9% and 10%
d) between 10% and 11%
e) more than 11%



20. Suppose $x = \log_{\ln 7} 7$, where *ln* denotes the natural logarithm. Which of the following inequalities is true?

a) $x < 0$ b) $0 < x < 1$ c) $1 < x < 7$ d) $7 < x < 10$ e) $10 < x$

21. A triangle in the coordinate plane is formed by the axes and the line whose equation is $3x + 5y = 30$. How many points (x, y) with integer coordinates lie inside the triangle?

a) 10 b) 16 c) 22 d) 23 e) 29

22. Each vertex of a regular nonagon (nine-sided polygon) is connected to the other vertices with line segments. How many of these segments lie inside the nonagon?

a) 14 b) 27 c) 36 d) 54 e) 72

23. Circle O is circumscribed about one square and inscribed inside another. What is the ratio of the area of the smaller square to that of the larger square?

a) 1/2 b) 1/3 c) 1/4 d) 3/8 e) $2/\pi$

24. Let $f(x)$ be the smallest non-negative number y such that $x - y$ is an integer. Compute $-2.2 + f(-2.2)$.

a) -3.0 b) -2.4 c) -1.4 d) 0 e) 0.8

25. If $\sin x = \cos 2x - 9/8$, then $\sin x =$

a) -1/4 b) 1/4 c) -1/16 d) 1/16 e) There is more than one possible value.

26. The average of all the numbers from 1 to 100 whose final digit is an odd prime number is

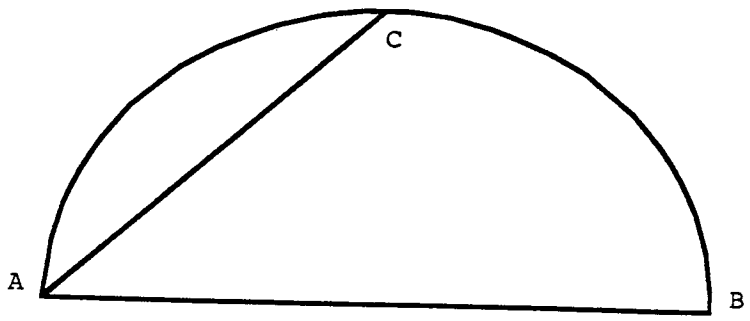
a) 45 b) between 45 and 50 c) 50 d) between 50 and 55 e) 55

27. Consider the points $A = (1, 3)$, $B = (4, 6)$, $C = (8, 2)$, and $D = (9, 3)$ in the coordinate plane. Suppose the polygonal line $ABCD$ is the graph of a function f. For any real number s, define $N(s)$ to be the number of solutions x of the equation $f(x) = s$. Then $N(2) + N(2.5) + N(3) =$

a) 2 b) 3 c) 5 d) 6 e) 7.5

28. Diameter AB of circle O has length 2. Point C is the midpoint of arc AB. Find the area of the curvilinear triangle ABC formed by arc BC and segments AB and AC.

a) $(\pi - 1)/2$
b) $(\pi + 1)/4$
c) $(\pi + 2)/4$
d) $(\pi + 3)/4$
e) None of these.



29. The periodic function $f(x) = \sin(x/2) + \sin(x/3)$ has period

a) 2π b) 3π c) 6π d) 12π e) 18π

30. If a is a real number such that $1 + \dfrac{1}{1+1/a} < \dfrac{8}{5}$, then

a) $-1 < a < 3/2$ b) $a > 5/2$ c) $a > 5/3$ d) $1/a < 3/5$ e) none of these follows

31. Five students were seated in a row. Phyllis sat next to Ed. Sue sat next to Sam. Ralph sat in the third seat from Sam. Sue sat in the third seat from Phyllis. There was a student sitting on the other side of Ed from Phyllis. The student's name was

a) Phyllis b) Sue c) Ralph d) Sam e) Not enough information

32. The lines with equations $y = 2x + 3$, $y = 6 - x$ and $y = 0$ meet at points A, B and C. Find the area of triangle ABC.

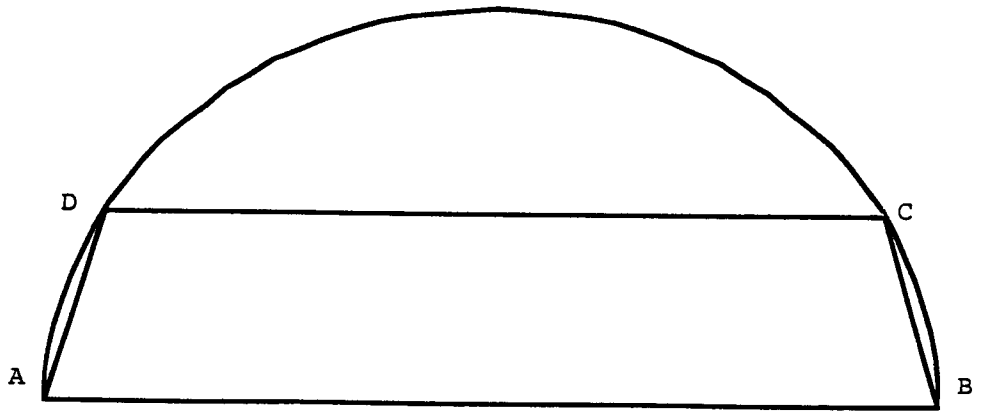
a) 37.5 b) 22.5 c) 20 d) 18.75 e) 11.25

33. How many two-digit numbers have the property that the product of the two digits exceeds 80% of the number?

a) 0 b) 1 c) 2 d) 3 e) 4

34. Trapezoid $ABCD$ is inscribed in a semicircle. Diameter AB has length 4, and sides AD and BC each have length 1. What is the length of side DC?

a) 11/4
b) 3
c) 13/4
d) 7/2
e) 15/4



35. A cube has vertices A, B, C, D at the bottom, vertices E, F, G, H at the top, and parallel edges AE, BF, CG and DH. Point M is the midpoint of edge BF. Which of the following lines intersect line DM?

a) BC b) FH c) AC d) AG e) EC

36. The roots of the equation $12x^3 + 24x^2 + 11x - 1 = 0$ are all

a) positive b) negative c) between -1 and 1
d) imaginary e) none of these statements is true

37. In isosceles triangle ABC, sides AB and AC are of the same length and each twice the length of side BC. What is the cosine of angle BAC?

a) 7/8 b) 1/2 c) $\sqrt{3}/2$ d) 3/4 e) $\sqrt{2}/2$

38. Prime numbers w, x, y, and z satisfy the equations $wxyz = 210$ and $w^2+2x^2+2y^2+2z^2 = 165$. Find w.

a) 2 b) 3 c) 5 d) 6 e) 7

39. Given the polynomial $f(x) = (3x + 5)(5x + 8)(8x + 13)$, which of the following numbers is positive?

a) $f(-5/3)$ b) $f(-7/4)$ c) $f(-8/5)$ d) $f(-18/11)$ e) $f(-21/13)$

40. The statement "If Mary lives in Michigan, then Mary knows her math" is logically equivalent to which of the following?

a) Mary doesn't know her math unless she lives in Michigan.
b) If Mary knows her math, then she lives in Michigan.
c) Mary knows her math only if she lives in Michigan.
d) Mary knows her math unless she doesn't live in Michigan.
e) Mary doesn't live in Michigan only if she doesn't know her math.

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