37TH MICHIGAN MATH PRIZE COMPETITION
Dec. 8, 1993
PART II SOLUTIONS

Problem 1.

a) (2 points) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ b) (2 points) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

c) (6 points) Consider an m by n matrix, with distinct entries, having saddle point at x. Assume y is also a saddle point. Let w be the entry in the row of x and the column of y, and let z be the entry in the column of x and the row of y. By definition of a saddle point, $z < x < w$, $w < y < z$, so $z < x < w < y < z$, a contradiction.

Problem 2.

a) (4 points) (14, 22), (18, 18) (8, 88), (12, 84)

b) (6 points)

$$\frac{7}{m} - \frac{11}{n} = 1 \qquad 11m + 7n = mn \qquad 7n = (n-11)m$$

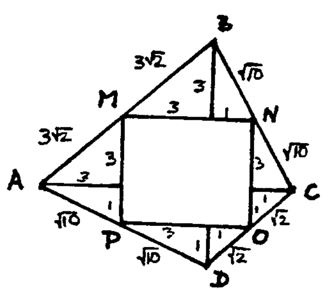
If $m = 7k$, then $7n = (n - 11)7k$, so $n = \frac{11k}{k-1}$, giving one solution for $k = 2$: $n = 22$, $m = 14$.

Also, if $n - 1 = 11j$, then $\frac{11(11j + 1)}{11j} = 11 + \frac{1}{j}$ giving for $j = 1$: $n = 12$, $m = 84$.
n is not an integer for any other j.

If $n - 11 = 7k$, then $7n = 7km$, $11 + 7k = km$, so $m = 7 + \frac{11}{k}$. For $k = 1$: $m = 18$, $n = 18$.
For $k = 11$: $m = 8$, $n = 88$.

m is not an integer for any other k.

Diagram for 3a)



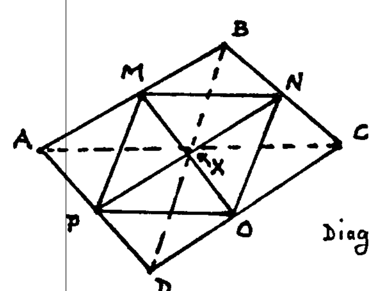


Diagram for 3b)

Problem 3.

a) (4 points)

$AB = 6\sqrt{2}$, $CD = 2\sqrt{2}$, ABCD is not a parallellogram.

b) (6 points)

(1) MNOP is a parallelogram, since MN & PO are parallel to AC, and MP & NO are parallel to BD.
(2) Therefore, diagonals MO and NP bisect each other.
(3) Since $MO \perp NP$ (given), then $\triangle MXN \cong \triangle OXN$, so MNOP is a rhombus.
(4) Using similar triangles, $AC = 1/2\ MN = 1/2\ MP = BD$.

Problem 4.

a) (2 points) $(3n, 4n, 5n)$, for n a positive integer, gives infinitely many pythagorean triples.

b) (4 points) Yes. $a^2 + b^2 = (b + 2)^2 = b^2 + 4b + 4$, so $a^2 = 4(b + 1)$. If a is a multiple of 4, say $a = 4n$, then b is found to be an odd integer, and the primitive triples are $(4n, 4n^2 - 1, 4n^2 + 3)$. Or, if $a^2 = 2n$ and $b + 1 = n^2$, we get $(2n, n^2 - 1, n^2 + 1)$.

c) (4 points) No. $a^2 + b^2 = (b + 3)^2 = b^2 + 6b + 9$, so $a^2 = 3(2b + 3)$. Clearly, a must be a multiple of 3, say, $a = 3n$. Then $2b = 3(n^2 - 1)$, so b must be a multiple of 3, and the triples are not primitive.

Problem 5.

a) (2 points) $s > 1$ when $y < 3$, say, $x = 1$, $y = 2$: in this example, $3x/2 = y/4 + 1/x = 3/y = 3/2$.

b) (4 points) Suppose that $3x/2 \geq 2$ and $3/y \geq 2$. Then we need to show that $y/4 + 1/x < 2$. With $x \geq 4/3$, then $1/x \leq 3/4$, and with $y \leq 3/2$, then $y/4 \leq 3/8$, so $y/4 + 1/x \leq 9/8 < 2$.

c) (4 points) Here are 3 different proofs.

(A) Using the reasoning of part b), Let M be the maximum value of s. Then $3x/2 \geq M$ and $3/y \geq M$, so $x \geq 2M/3$ and $y \leq 3/M$. Hence $y/4 + 1/x \leq 3/4M + 3/2M = 9/4M$, which is a maximum when $9/4M = M$, giving $M = 3/2$. (Note: s is maximum when $x = 1$ and $y = 2$.)

(B) It is reasonable to assume that s is maximal when all three numbers are equal. Solve the system $3x/2 = y/4 + 1/x = 3/y$ to get $s = 3/2$. Prove that $s \leq 3/2$. Suppose that $s > 3/2$. Then $3x/2 > 3/2$, so $x > 1$, and $3/y > 3/2$, so $y < 2$. Then $y/4 + 1/x < 1/2 + 1 = 3/2$, which contradicts $s > 3/2$.

(C) Let $p = 3x/2$, $q = 3/y$, and $f(p,q) = 3/2p + 3/4q$. Express the upper right half plane as the union of pairs of rays emanating from (p,p): as you move up along r_1 from (p,p), p is constant, q increases, and $f(p,q)$ decreases. So $s = \min\{p, f(p,q)\}$ on r_1, and the maximum of s on r_1 occurs at (p,p). Similarly, the maximum of s on r_2 occurs at (p,p). So we need only check s on the line $p = q$. Here, $s = \min\{p, f(p,p)\} = \min\{p, 9/4p\}$. As p increases, $9/4p$ decreases so that max s occurs when $p = 9/4p$, or $p = 3/2$. Thus, max $s = 3/2$ when $p = q = 3/2$.