MMPC CODE NUMBER

THIRTY-SEVENTH ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 8, 1993

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page, clearly write the **problem number**. If you use additional paper for your answer, **check the appropriate box** and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. If you are unable to solve a particular problem, partial credit will be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

7. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference materials, computational aids, or any other aids is likewise prohibited. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. It is not necessary to return scratch paper on which routine numerical calculations have been made.

8. You may now open the test booklet.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

1. A matrix is a rectangular array of numbers. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

are 2×2 matrices. A *saddle point* in a matrix is an entry which is simultaneously the smallest number in its row and the largest number in its column.

a. (2 points) Write down a 2×2 matrix which has a saddle point, and indicate which entry is the saddle point.

b. (2 points) Write down a 2×2 matrix which has no saddle point.

c. (6 points) Prove that a matrix of any size, all of whose entries are distinct, can have at most one saddle point.

Check this box if your solution is continued on inside back cover (page 7) ☐
Check this box if your solution is continued on inserted additional paper ☐

2. a. (4 points) Find four different pairs of positive integers satisfying the equation
$$\frac{7}{m} + \frac{11}{n} = 1 .$$
   b. (6 points) Prove that the solutions you have found in part (a) are all possible pairs of positive integers satisfying the equation $\frac{7}{m} + \frac{11}{n} = 1$.

Check this box if your solution is continued on inside back cover (page 7) ☐

Check this box if your solution is continued on inserted additional paper ☐

3. Let $ABCD$ be a quadrilateral, and let points M, N, O, P be the respective midpoints of sides AB, BC, CD, DA.

   a. (4 points) Show, by example, that it is possible that $ABCD$ is not a parallelogram, but $MNOP$ is a square. Be sure to *prove* that your construction satisfies all given conditions.

   b. (6 points) Suppose that MO is perpendicular to NP. Prove that $AC = BD$.

Check this box if your solution is continued on inside back cover (page 7) ☐

Check this box if your solution is continued on inserted additional paper ☐

4. A Pythagorean triple is an ordered collection of three positive integers (a, b, c) satisfying the relation $a^2 + b^2 = c^2$. We say that (a, b, c) is a *primitive* Pythagorean triple if 1 is the only common factor of a, b, and c.

   a. (2 points) Decide, with proof, if there are infinitely many Pythagorean triples.

   b. (4 points) Decide, with proof, if there are infinitely many *primitive* Pythagorean triples of the form (a, b, c) where $c = b + 2$.

   c. (4 points) Decide, with proof, if there are infinitely many *primitive* Pythagorean triples of the form (a, b, c) where $c = b + 3$.

Check this box if your solution is continued on inside back cover (page 7) ☐
Check this box if your solution is continued on inserted additional paper ☐

5. Let x and y be positive real numbers and let s be the smallest among the numbers

$$\frac{3x}{2},\ \frac{y}{4}+\frac{1}{x},\ \text{and}\ \frac{3}{y}.$$

a. (2 points) Find an example giving $s > 1$.

b. (4 points) Prove that for any positive x and y, $s < 2$.

c. (4 points) Find, with proof, the largest possible value of s.

Check this box if your solution is continued on inside back cover (page 7) ☐

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Continuation of solution to problem number ☐

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America

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The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.