THIRTY-SEVENTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 13, 1993

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. **Check to insure that your six-digit code number has been recorded correctly.** Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "**e)** none of the others". If you believe none of the first four alternatives to be correct, mark **e)** in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You many now open the test booklet and begin.

1. Let c represent the number of clerks and let a represent the number of accountants at the Ajax Company. Which expression reflects the fact that there are more than twice as many clerks as accountants?

   a) $c > 2 + a$ b) $c > 2a$ c) $2a > c$ d) $a > 2c$ e) $2c > a$

2. If the sum of four consecutive integers is 34, what is their product?

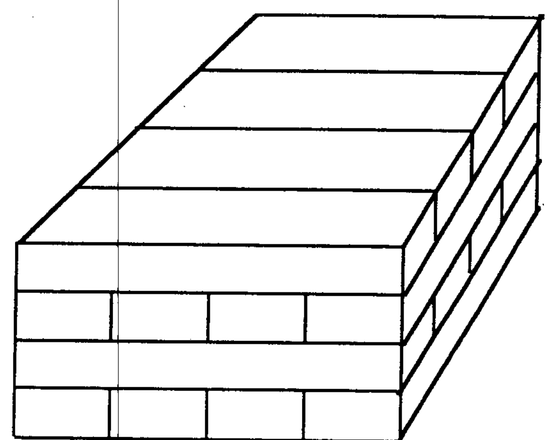
   a) 504 b) 720 c) 3024 d) 5040 e) 7920

3. A stack of sixteen 1″ x 2″ x 8″ blocks are arranged in four one-inch layers, as shown. Find the number of blocks which make contact with at least ten other blocks.

   a) zero b) two

   c) four d) six

   e) eight



4. A movie-theater charged $6.00 per ticket. In order to attract more customers, the price of the ticket was decreased, resulting in 25% more sold tickets. The revenue increased by 12.5%. What was the new ticket price?

   a) $4.50 b) $5.00 c) $5.40 d) $5.50 e) $5.60

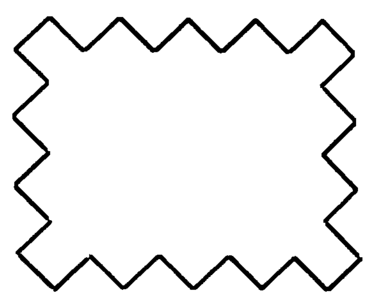
5. One can change the value of the expression $3 \cdot 2 + 8 / 2 + 2$ by inserting a set of parentheses. What is the largest value one can obtain?

   a) 12 b) 17 c) 20 d) 24 e) 30

6. In the figure to the right, each segment is 1 cm and each angle is a right angle. What is the area inside the figure, in cm^2?

   a) 12.5 b) 20 c) 22

   d) 32 e) 40



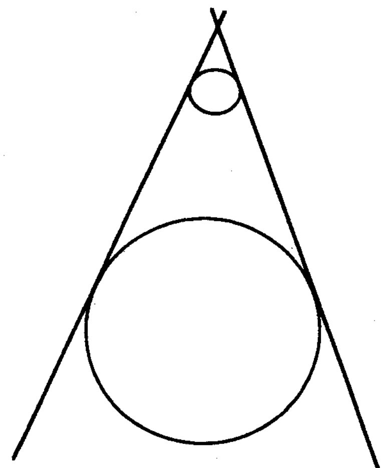
7. Given a positive number x, which of the following is not equal to $x^{-1/6}$?

a) $\frac{1}{\sqrt[3]{x^{0.5}}}$ b) $\frac{1}{\sqrt[6]{x}}$ c) $\frac{1}{\sqrt{\sqrt[3]{x}}}$ d) $\frac{\sqrt[6]{x^5}}{x}$ e) $\sqrt{\frac{1}{x^3}}$

8. If $i^2 = -1$, then $(1 + i)^{20} =$

a) 2   b) $20 - 20i$ c) $20 + 20i$ d) 1024   e) -1024

9. The radii of the two given circles are 1 and 5, and the distance between their centers is 10. What is the distance between the point where their common external tangents meet and the center of the smaller circle?



a) 2.0   b) 2.25   c) 2.3

d) 2.4   e) 2.5

10. If $b > 0$, solve the equation $\log_b (x - 1) + \log_b (x + 1) = 1$ for x in terms of b.

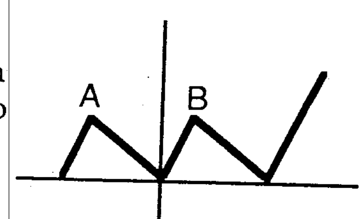
a) $\sqrt{b}$ b) $\pm\sqrt{b - 1}$ c) $\pm\sqrt{1 + b}$ d) $\sqrt{b - 1}$ e) $\sqrt{1 + b}$

11. Consider the quadratic functions $f(x) = 2x^2 + bx - 5$ for various choices of b. If b is positive, then the vertex of the graph lies in quadrant ...

a) I   b) II   c) III   d) IV

e) Cannot be determined from given information.

12. If the graph of $y = f(x)$ is shown to the right, then which of the following numbers cannot be equal to the number of solutions of the equation $f(x) = 1$? (Points A and B have the same y-coordinate.)



a) 0   b) 1   c) 2   d) 3   e) 5

13. Which of the following is equal to $\sqrt{5 + \sqrt{24}}$?

a) $\sqrt{3} + \sqrt{2}$ b) $\sqrt{3} - \sqrt{2}$ c) $2 + \sqrt{3}$ d) $3 + \sqrt{2}$ e) $\sqrt{5} + \sqrt{24}$

14. A rhombus has sides of length 1, and an angle of 60°. What is the radius of its inscribed circle?

a) $\frac{\sqrt{3}}{4}$ b) $\frac{1}{4}$ c) $\frac{\sqrt{3}}{2}$ d) $\frac{1}{2}$ e) None of the others

15. If $a_0 = 1$ and for $n > 0$, $a_n = a_0 + a_1 + \ldots + a_{n-1}$, then $a_{100} =$

a) $2^{99} - 1$ b) 2^{99} c) $2^{100} - 1$ d) 2^{100}
e) None of the others

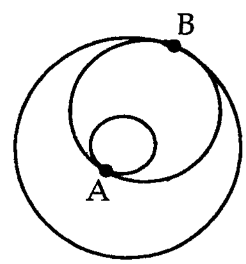
16. A side of one equilateral triangle is congruent to an altitude of another equilateral triangle. What is a ratio of the areas of the triangles?

a) $\frac{\sqrt{3}}{2}$ b) $\frac{\sqrt{2}}{3}$ c) $\frac{\sqrt{3}}{4}$ d) $\frac{2}{3}$ e) $\frac{3}{4}$

17. A real number x is a solution to the equation $\sin(19\pi x) + \cos(93\pi x) = 1$. Which of the following numbers is another solution to this equation?

a) $x + 1$ b) $x - 2$ c) $x + \pi$ d) $x - 2\pi$ e) none of the others

18. Three circles are arranged so that two are concentric and the third is tangent to them both, at points A and B. If the largest two circles have radii 10 and 7, respectively, find the area of the smallest circle as a percentage of the largest circle.



a) 9% b) 12% c) 14% d) 16% e) 20%

19. A carpenter is building a deck. He hires two assistants who work 60% as fast as he works. If all three work together on the deck, they should finish it in what fraction of the time that he would take working alone?

a) 5/11 b) 11/15 c) 3/13 d) 5/13 e) 3/5

20. Which equation expresses y as a function of x, given that the graph of $\log_{10} y$ vs $\log_{10} x$ is a straight line with slope m. (b and c are constants)

a) $y = mx + b$ b) $y = x^m + b$ c) $y = cx^m$ d) $y = c\cdot 10^{mx}$ e) $y = m\cdot 10^x$

21. If two real numbers have a sum of 8 and a product of 14, what is the sum of their reciprocals?

a) $\frac{1}{8}$ b) 8 c) $\frac{4}{7}$ d) $\frac{7}{4}$ e) $\frac{4-\sqrt{2}}{4+\sqrt{2}}$

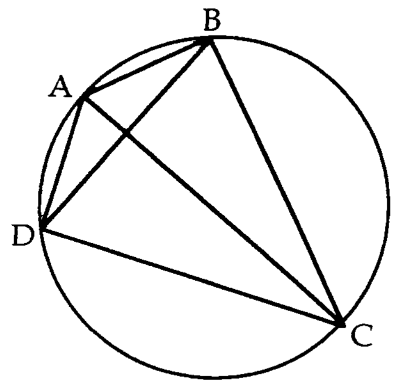
22. What is the last digit of the product of all odd numbers from 1 to 999 ?

a) 1 b) 3 c) 5 d) 6 e) 9

23. Given real numbers x, y, z, and w such that $|x - y| = 4$, $|y - z| = 5$, and $|z - w| = 7$, what is the smallest possible value of $|x - w|$?

a) 0 b) 2 c) 3 d) 6 e) 16

24. Find the area of quadrilateral ABCD which is inscribed in a circle of radius 10, given that the diagonal AC is a diameter of the circle and the diagonal BD is perpendicular to AC at a point two inches from the circle.



a) 60 b) 80 c) 90
d) 120 e) 150

25. Which number below will produce a *whole* number when multiplied by $\sqrt[3]{25} + \sqrt[3]{10} + \sqrt[3]{4}$?

a) $(\sqrt[3]{25} + \sqrt[3]{10} + \sqrt[3]{4})^2$ b) $(\sqrt[3]{25} - \sqrt[3]{10} + \sqrt[3]{4})^2$ c) $\sqrt[3]{5} + \sqrt[3]{2}$ d) $\sqrt[3]{5} - \sqrt[3]{2}$
e) No such number exists.

26. Radii OA, OB, and OC divide the circle into three congruent parts. Let D be a point on the arc AB such that arc AD has half the measure of arc DB. Find the measure of $\angle OCD$.

a) 5° b) 10° c) 15° d) 20° e) 25°

27. A triangular region has its vertices at the points (0, 0), (0, 1), and (1, 0). Find the number m such that the vertical line $x = m$ separates this region into two subregions of equal area.

a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $1 - \frac{\sqrt{3}}{2}$ d) $\frac{\sqrt{3}}{2}$ e) $1 - \frac{\sqrt{2}}{2}$

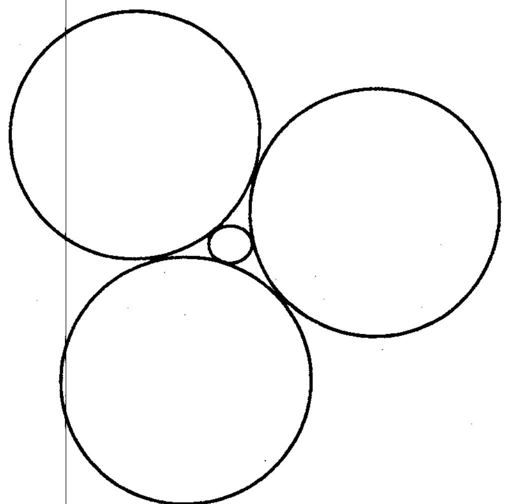
28. AD and BE are altitudes of ΔABC. If AC = 10, BC = 8, and AD = 5, then BE = ...

a) 4 b) 5 c) 6 d) 9
e) Depends on whether $\angle ACB$ is acute, right, or obtuse.

29. If $\tan x = 0.5$, then which of the following must be true ?

a) $\sin x = 2 \cos x$ b) $\sin x > \cos x$ c) $\sin x < \cos x$
d) $|\sin x| > |\cos x|$ e) $|\sin x| < |\cos x|$

30. In the accompanying figure, find the radius of the inner circle which is tangent to all three outer circles, each having radius 1.



a) $\frac{2\sqrt{3}}{3} - 1$ b) $\frac{3\sqrt{2}}{2} - 2$

c) $\frac{\sqrt{3} - 1}{3}$ d) $\frac{\sqrt{2} - 1}{3}$

e) $\frac{1}{9}$

31. $2^{\log_7 3 / \log_7 2} =$

a) 2 b) 3 c) 7 d) $\frac{3}{2}$ e) $\frac{7}{3}$

32. Suppose $a, b, c,$ and d are positive numbers and $\frac{a}{c} < \frac{b}{d}$. What fraction of the way from $\frac{a}{c}$ to $\frac{b}{d}$ is $\frac{a+b}{c+d}$?

a) $\frac{a}{c+d}$ b) $\frac{b}{c+d}$ c) $\frac{c}{c+d}$ d) $\frac{d}{c+d}$

e) $\frac{a+b}{c+d}$ isn't necessarily between $\frac{a}{c}$ and $\frac{b}{d}$.

33. If you write down the whole numbers from 1 to 1000 in the usual order, what is the 1000^{th} digit you will write?

a) 1 b) 3 c) 5 d) 7 e) 9

34. One chooses randomly an integer n such that $1 \leq n \leq 100$. What is the probability that the fraction $n/162$ is irreducible?

a) .162 b) .17 c) .33 d) .81 e) .83

35. In triangle ABC, side AC is of length 1, angle A is 30°, and angle B is 15°. What is the length of side AB?

a) $\frac{\sqrt{2}}{2}$ b) $\sqrt{6} - \sqrt{2}$ c) $\sqrt{6} + \sqrt{2}$ d) $\sqrt{3} - 1$ e) $\sqrt{3} + 1$

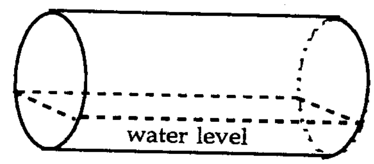
36. Compute

$$\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^7+\left(\frac{1}{2}\right)^8+\left(\frac{1}{2}\right)^{10}+\left(\frac{1}{2}\right)^{11}+\cdots$$

where the exponents are all positive integers, except multiples of 3.

a) $\frac{1}{3}$ b) $\frac{2}{3}$ c) $\frac{219}{256}$ d) $\frac{6}{7}$ e) $\frac{13}{7}$

37. A right circular cylindrical tank with diameter 4 ft and length 8 ft is lying on its side, as shown. In this position, the tank contains water, and the depth of the water at its deepest point is 1 ft. When the cylinder is turned upright, the depth of the water will be ...



a) less than 2 ft b) 2 ft c) between 2 ft and 3 ft
d) 3 ft e) more than 3 ft

38. Find the largest integer N such that $\frac{1}{1000}+\frac{2}{1000}+\frac{3}{1000}+\cdots+\frac{N}{1000}<1.$

a) 44 b) 66 c) 110 d) 500 e) 999

39. The smallest positive integer which gives a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, a remainder of 3 when divided by 4, a remainder of 4 when divided by 5, and a remainder of 5 when divided by 6, is ...

a) less than 50 b) between 50 and 100 c) between 100 and 200
d) between 200 and 400 e) greater than 400

40. A geometry student classified all triangles as follows: A triangle is called *alphish* if it is either a right triangle or an isosceles triangle, but not both. A triangle is called *betish* if it is either an isosceles triangle or has a side longer than two inches, but not both. A triangle is called *gammish* if it is either alphish or betish, but not both. Which of the following is true?

a) Every right triangle is gammish.
b) Every gammish triangle is right.
c) Every gammish triangle is either right or has a side longer than two inches.
d) Every triangle which is both right and has a side longer than two inches is gammish.
e) None of the above are true.

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ACKNOWLEDGEMENTS

The following individuals, corporations and professional organizations have contributed generously to this competition:

Addison-Wesley Publishing Co.
Ford Motor Company
Jerome J. Kohel
John Wiley & Sons
Kuhlman Corporation
Lawrence Technological University
Matilda R. Wilson Fund
Monroe Auto Equipment
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