36TH MICHIGAN MATH PRIZE COMPETITION
Dec. 9, 1992
PART II SOLUTIONS

Problem 1.
a) (4 points) The 5 vowels divide the list into (at most) 6 blocks of consonants. If each block had length only (at most) 3, there would be at most only 18 consonants.

Alternate: Break the list (minus the last two letters) into 6 groups of 4 consecutive letters each. One must lack a vowel.

b) (1 point) bcdfAghjkElmnpIqrstOvwxyUz

c) (5 points) The vowels divide the circle into only 5 blocks, so 4 consonants per block give at most only 20 consonants.

Alternate: Remove one of the vowels, and break the resulting list of 25 into 5 blocks of 5; one must lack a vowel.

Problem 2.
a) (3 points) There are six positions to fill. With 2 in position 1, the remaining five positions are filled from the range of 3 to 10. There are $\binom{8}{5}$ ways to choose these. There are $\binom{10}{6}$ possible choices altogether. The probability P(2, 1,6,10) is $\binom{8}{5}/\binom{10}{6} = 4/15$.

Alternate:For 2 to occur in position 1 means that 1 was not chosen but 2 was. The probability of *not* choosing 1 is 4/10. Similarly, the conditional probability of choosing 2, given that 1 was not chosen is 6/9. Multiplying, we get 4/15. Also, one could use the probability of choosing 2 (namely 6/10) times the conditional probability of not choosing 1 given that 2 is chosen (namely 4/9) to get the answer 4/15.

b) (7 points) For i to occur in position r, we must have chosen r - 1 of the of the i - 1 numbers below i, then i itself, and k-r of the n - i numbers above i. So $P(i,r,k,n) = \binom{i-1}{r-1}\binom{1}{1}\binom{n-i}{k-r}/\binom{n}{k}$

Problem 3.
a) (2 points) The simplest solution is $x^4 + x(x^2 - 1) = x^4 + x^3 - x$. For any solution, the odd degree terms must be a non-zero constant multiple of $x^3 - x$, while the even degree terms are arbitrary, except that x^4 must have a non-zero coefficient.

b) (2 points) The simplest solution is $x(x^2 -1)(x^2 - 4) = x^5 - 5x^3 + 4x$. Any solution is a non-zero constant multiple of this, plus arbitrary terms of even degree.

c) (6 points) The polynomial $R(x) - R(-x)$ has degree at most 6 (in fact, at most 5, since the even degree terms cancel) and vanishes and vanishes at seven points, namely $0, \pm 1, \pm 2. \pm 3$.
But a non-zero polynomial of degree n has at most n zeros, so $R(x) - R(-x) = 0$.

Alternate: (a) If the undetermined coefficentsof a fourth degree polynomial are a, b, c, d, e then $b + d = 0$ for $P(-1) = P(1)$ and $4b + d \neq 0$ for $P(-2) \neq P(2)$.

(b) If the undetermined coefficicients are a, b, c, d, e, f then $a + c + e = 0$ and $16a + 4c + e = 0$; $81a + 9c + e \neq 0$.

(c) Write R(x) as a 6th degree polynomial with undetermined coefficients a, b, c, d, e, f. Applying the conditions gives a 3rd order homogeneous system of equations for the coefficients of the odd powers of x (see below), which has only the trivial solution. So R(x) is even.

$$b + d + f = 0, \quad 16b + 4d + f = 0, \quad 81b + 9d + f = 0.$$

Problem 4.

a) (3 points) -2, -1, 0, 1, 2 yields only 7 different sums, so N = 7.

b) (3 points) Any sequence of powers of an integer will do: e.g., 1, 2, 4, 8, 16 or 1, 10, 10^2, 10^3, 10^4. All possible sums are distinct, so N = 26.

c) (4 points) The fact that all sums in b) are distinct shows b) is optimal. To show a) is optimal, consider the ordered set of 5 numbers $a < b < c < d < e$. Of all pairwise sums, there are at least the distinct set $a + b < a + c < a + d < a + e$.

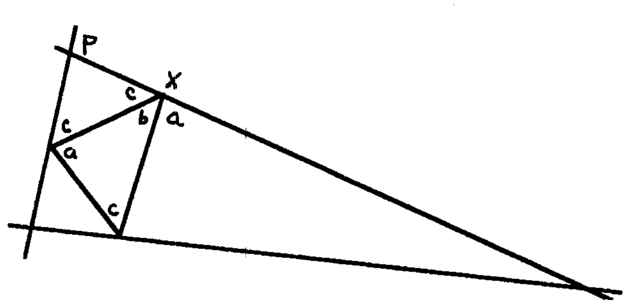
Problem 5.

(A) See diagram (A)) For an acute triangle: Use each side of the given triangle as the base of an isosceles triangle, having base angles equal to the angle of the vertex opposite that side. Note that the sum of the three angles meeting at any original vertex X is 180°, since they are the sum of the three angles of the triangle. This means that the sides of the two new triangles meeting there lie on a straight line. Construct circles whose centers are the apex of the new triangles (e.g., PX). The circles which meet at any original vertex are tangent there because they are both perpendicular to the line constructed at that vertex.
For an obtuse triangle: The same argument works, except that the isosceles triangle constructed on the side opposite the obtuse angle must overlap the given triangle rather than lying outside of it.

(B) (See diagram (B)) Let O be the circumcircle of the given triangle. The tangents to any two points X and Y on O (not on a diameter) will intersect at point P. Since tangents from an external point P create congruent segments PX and PY, draw the circle Q with center P and radius segments of PX and PY.The circle Q is perpendicular to the circle O at X and Y. Repeat this procedure for each pair of vertices. The two circles passing through each vertex will be tangent there because they are both perpendicular to the circumcircle.

Alternate: If the desired circles exist, then the angles they make with the edges at the vertices are subject to enough linear constraints to determine them uniquely. Then use the angles to determine the circles.

(A)



(B)