MMPC CODE NUMBER

THIRTY-SIXTH ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

Part II

Wednesday, December 9, 1992

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**

2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all the problems completely. Look over all the problems and work first on those that interest you the most.

4. Each problem is on a separate page. If possible, you should show all of your work on that page. If there is not enough room, you may continue your solution on the inside back cover (page 7) or on additional paper inserted into the examination booklet. Be certain to **check the appropriate box** to report where your continuation occurs. On the continuation page, clearly write the **problem number**. If you use additional paper for your answer, **check the appropriate box** and write your **identification number** and the **problem number** in the upper right-hand corner of each additional sheet.

5. If you are unable to solve a particular problem, partial credit will be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate an idea of how to solve the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

7. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference materials, computational aids, or any other aids is likewise prohibited. When the supervisor announces that the 100 minutes are over, please cease work immediately and insert all significant extra paper into the test booklet. It is not necessary to return scratch paper on which routine numerical calculations have been made.

8. You may now open the test booklet.

PROBLEM	SCORE
1	
2	
3	
4	
5	
TOTAL	

Problem 1.

The English alphabet consists of 21 consonants and 5 vowels. (We count y as a consonant.)

(a) (4 points) Suppose that all the letters are listed in an arbitrary order. Prove that there must be 4 consecutive consonants.

(b) (1 point) Give a list to show that there need not be 5 consecutive consonants.

(c) (5 points) Suppose that all the letters are arranged in a circle. Prove that there must be 5 consecutive consonants.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

or on inserted additional paper ☐

Problem 2.

From the set $\{1, 2, 3, \ldots, n\}$, k distinct integers are selected at random and arranged in numerical order (lowest to highest). Let $P(i, r, k, n)$ denote the probability that integer i is in position r. For example, observe that $P(1, 2, k, n) = 0$.

(a) (3 points) Compute $P(2, 1, 6, 10)$.

(b) (7 points) Find a general formula for $P(i, r, k, n)$.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

or on inserted additional paper ☐

Problem 3.
(a) (2 points) Write down a fourth degree polynomial $P(x)$ such that $P(1) = P(-1)$ but $P(2) \neq P(-2)$.
(b) (2 points) Write down a fifth degree polynomial $Q(x)$ such that $Q(1) = Q(-1)$ and $Q(2) = Q(-2)$ but $Q(3) \neq Q(-3)$.
(c) (6 points) Prove that, if a sixth degree polynomial $R(x)$ satisfies $R(1) = R(-1)$, $R(2) = R(-2)$, and $R(3) = R(-3)$, then $R(x) = R(-x)$ for all x.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

or on inserted additional paper ☐

Problem 4.

Given five distinct real numbers, one can compute the sums of any two, any three, any four, and all five numbers and then count the number N of *distinct* values among these sums.

(a) (3 points) Give an example of five numbers yielding the smallest possible value of N. What is this value?

(b) (3 points) Give an example of five numbers yielding the largest possible value of N. What is this value?

(c) (4 points) Prove that the values of N you obtained in (a) and (b) are the smallest and largest possible ones.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

or on inserted additional paper ☐

Problem 5.

Let $A_1A_2A_3$ be a triangle which is not a right triangle. Prove that there exist circles C_1, C_2, and C_3 such that C_2 is tangent to C_3 at A_1, C_3 is tangent to C_1 at A_2, and C_1 is tangent to C_2 at A_3.

Check the appropriate box if your solution is continued on inside back cover (page 7) ☐

or on inserted additional paper ☐

Continuation of solution of problem number ☐

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Assocation of America

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The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.