THIRTY-FIFTH ANNUAL MICHIGAN MATH PRIZE COMPETITION

SOLUTION KEY

The answer key to Part One of the Thirty-Fifth annual Michigan Math Prize Competition, which was given on October 9, 1991 is as follows:

1. E	11. A	21. A	31. D
2. A	12. B	22. A	32. C
3. E	13. B	23. B	33. E
4. B	14. D	24. B	34. E
5. E	15. C	25. B	35. B
6. C	16. A	26. D	36. E
7. D	17. A	27. C	37. C
8. C	18. D	28. B	38. B
9. B	19. E	29. A	39. D
10. D	20. C	30. E	40. D

A few comments and hints about some of the problems follow.

10. There was an implicit assumption that AC is perpendicular to OB, which had to be made in order to get one of the possible answers. All answers given were specific. Draw the radius OC, to find the ratio $(\pi/3)/(\pi/6)=2$.

11. Area(moat + island) = Area(moat) + Area(island) = 2 Area(island). Thus $w^2 + 100w + 2500 = 0$ where w is the width of the moat.

14. For one section of the exam, ${}_5C_3 = 10$. So for both sections the total number of ways is, $10 \times 10 = 100$.

17. $1000 = 2^3\,5^3$ so the list of divisors is $1, 2, 5, 2^2, 5^2, 2^3, 5^3, 2\text{x}5, 2\text{x}5^2, 2\text{x}5^3, 2^2\text{x}5, 2^2\text{x}5^2, 2^2\text{x}5^3, 2^3\text{x}5, 2^3\text{x}5^2, 2^3\text{x}5^3$. The product is $2^{24}\text{x}5^{24} = 10^{24}$.

18. Since $2^{10} = 1024$, and 2^{-10} is very small, $2^{10} - 2^{-10} < 1024$, so $5 < x < 20$

19. Factor into $(3u - 4v)^2 - 4w^2$, and factor again.

23.The graph has even symmetry in both x and y. The equation must have even powers, but no even roots, since it must be satisfied by negative values of both x and y.

26. The middle term is $(6x)^3(1/2x)^3$. Use Pascal's triangle: [1 6 15 20 ...] to give $-20\,(6^3)/(2^3) = -540$.

27. A nice use of the formula for the sum of the first n integers If A = 4 ft is the distance covered in the first second, then in 11 seconds,

$$A + [A + (1)5] + [A + (2)5] + ... + [A + (10)5] = 11A + 5[1+2+...+10] = 11(4) + 5[10(11)/2] = 319$$

29. Diagram, page 2.

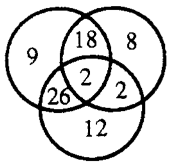
30. - 33. are for people who like triangles. Diagrams are on page 2.

36. While this problem can be done rather quickly by educated trial and error (eliminating any number but 1 for a and trying 5 or 6 for b), another way is to observe

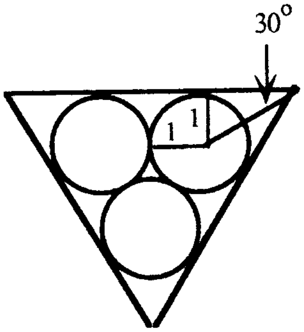
$11 \equiv 4 \bmod 7$ so $49a - 7b + c \equiv 4 \bmod 7$, giving $c = 4$. Then $7(1 + b) = 49a$, so $a = 1$ and $b = 6$.

37. With 7 distinct letters, there are ${}_7P_3 = 7(6)(5) = 210$ 3-letter distinguishable sequences. If I is to be used twice, there are 18 distinguishable sequences: 3 possible positions for the first I, then 6 choices for another letter. The second I goes in the remaining spot.

29.

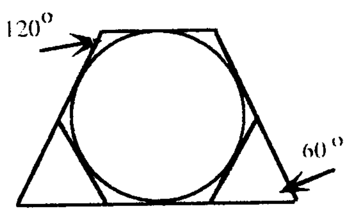


30.



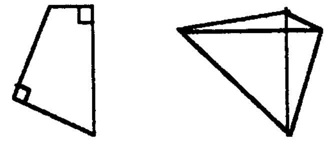
Use a 30-60-90- right triangle to get half the side length of $1 + \sqrt{3}$.

32.



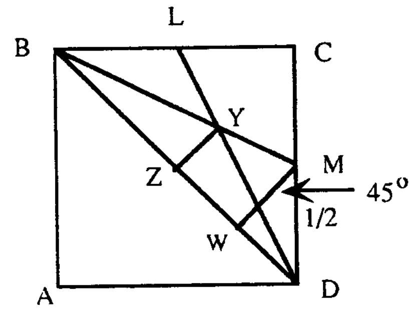
The ratio of top base to bottom base is clearly 1/3

38.



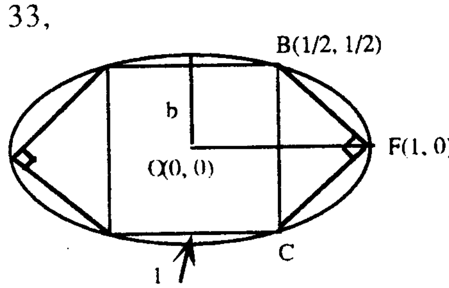
These diagrams show why II and III are false.

31.



The area of triangle BZY is 1/4 the desired area. Triangle BZY is similar to triangle BWM. $|BZ| = (1/2)\sqrt{2}$ and $|WM| = (1/4)\sqrt{2}$, giving $|ZY| = (1/6)\sqrt{2}$.

33,



Angle CBF = 45^o, so $|OF| = 1/2 + 1/2 = 1$. Substitute the coordinates of B into the equation for an ellipse, semimajor axis $a = 1$: so $b^2 = 1/3$