October 25, 1990

Dear MMPC Supervisor:

The answer key to Part One of the Thirty-Fourth annual Michigan Math Prize Competition, which was given on October 10, 1990 is as follows:

1. A	11. E	21. C	31. D
2. B	12. C	22. B	32. D
3. C	13. E	23. A	33. D
4. E	14. E	24. C	34. D
5. C	15. D	25. A	35. B
6. B	16. B	26. D	36. C
7. A	17. B	27. B	37. D
8. B	18. E	28. E	38. A
9. C	19. E	29. E	39. D
10. B	20. B	30. C	40. E

A few comments regarding some of the questions that seemed to give many of the students some difficulty follow:

2. $x^4 + y^4 = x^4 + 2x^2y^2 + y^4 - 2x^2y^2 = (x^2 + y^2 + \sqrt{2}\,xy)(x^2 + y^2 - \sqrt{2}\,xy)$

6. There are 31 squares less than 1000 and 44 squares less than 2000, thus 44-31 = 13 between .

9. Call the strips 1,2,3,4, and 5 from the top. Strip 1 can be colored with any of the 3 colors. Once a color is chosen for strip 1, only 2 choices are allowable for strip 2, once that is made, 2 are allowable for strip 3, etc., thus 3x2x2x2x2 = 48 colorings are possible.

10. f(x) is defined for only non-positive numbers, hence the value of $f^{-1}(x)$ must be non-positive. The value of f(x) is non-negative, hence the domain of $f^{-1}(x)$ must be non-negative.

13. Remember to count distance up **and** down. Distance = 10 + 5 + 5 + 2.5 + 2.5 + ... =
(10 + 5 + 2.5 + ...) + (5 + 2.5 + 1.25 + ...) = 20 + 10 = 30.

19. Sin of *any number* is less than $\sqrt{3}$.

20. 19^2 ends with the digit 1 in the units place, so equals 5k + 1. $19^{90} = (5k + 1)^{45}$, which is a sum of terms each divisible by 5 except for the last term, which is 1^{45}.

22. $\text{Log}_5 8x + 2\,\text{Log}_5 x = \text{Log}_5 8x^3$. Thus $8x^3 = 5^3$. The solution is x = 2.5.

23. Extend BO to E so that BE is a diameter. EC=3, CB=1, and AC = $\sqrt{5}$, using Pythagorus. You can use EC x CB = AC x CD to obtain the result.

24. If (-2)(-3) = -6, and (-2)(3) = -6, then (-2)((-3)+(3)) = (-2)(0) = 0, but
(-2)(3) + (-2)(-3) would = (-6) + (-6) = -12 , rather than 0.

25. Draw the line joining the centers of the 2 circles, OO'. Draw radii OA and O'B. Call the point of intersection of OO' with AB, C. Then triangles OAC and O'BC are similar right triangles . Since OA = 4 and O'B = 2, it follows that OC = 8 and O'C = 4. By Pythagorus, CB= $2\sqrt{3}$ and AC = $4\sqrt{3}$.

30. Since $\sqrt{x}$ is involved in the definition of f, f(x) is not defined for x<0, thus (a) is eliminated. Since $\sqrt{\sqrt{x} - x}$ is involved in the definition of f, and $\sqrt{x} - x$ is < 0 if x > 1, (d) is eliminated. Since $2\sqrt{\sqrt{x} - x} - 1$ is the denominator of f(x), and $2\sqrt{\sqrt{x} - x} - 1 =$ 0 when x = .25, (b) is eliminated. However if 0.5 < x < 0.8, $2\sqrt{\sqrt{x} - x} - 1$ is a nice negative number, so (c) is correct. (The largest value of $\sqrt{x} - x$ is .25 when x = .25, which can be seen by completing the square)

31. There are 216 equally likely outcomes. There are 4 (123,234,345,456) *combinations* which are consecutive, and 3! or 6 outcomes for each combination. Thus 24 out of 216, or $\frac{1}{9}$ is the correct answer.

32. Arc ABC = 60 degrees, so major arc AC is 300 degrees.

33. The top and bottom planes divide space into 3 regions. Each of these regions is subdivided by the remaining 4 planes into 9 sub-regions (think of a tic-tac-toe board). So there are 3x9 = 27.

34. You can use a symmetry argument to eliminate all but (b) and (d) by the fact that if (x,y) is a solution, then so is (-x,-y). However, if x = 1, and $-1 \le y \le 0$, $|x+y| = 1+y$, and $|y| = -y$, thus (x,y) is a solution, (Similarly for x = -1, $0 \le y \le 1$). So (d) is the correct solution.

35. The condition is equivalent to $-1 < \log_4|x| < 1$, which is equivalent to $\frac{1}{4} < |x| < 4$, which is equivalent to (b).

37. There are 25 equally likely choices for k and m, those for which $k^2 - 4m < 0$ satisfy the condition. A table shows that there are 13 cases where $k^2 - 4m < 0$.

38. $x^3 + y^3 = (x + y)(x^2 - xy + y^2) = (x + y)(x^2 + 2xy + y^2 - 3xy)$
$= (x + y)(\{x + y\}^2 - 3xy) = 6(36 - 21) = 6(15) = 90.$

40. The car traveled 30 km. in 50 min. It then traveled at 1 km. per min., so x km. in x min. So 30 + x = .8(50 + x), since 48 km./hr = 0.8 km/min. The result follows.