THIRTY-FOURTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by
The Michigan Section of the Mathematical Association of America

Part I

October 10, 1990

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. **Check to insure that your six-digit code number has been recorded correctly.** Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "**e)** none of these" or "**e)** none of the above". If you believe none of the first four alternatives to be correct, mark **e)** in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You many now open the test booklet and begin.

1. The expression $\frac{1+2i}{2-i}$ equals

a) i b) - i c) $\frac{4+3i}{5}$ d) $\frac{4+5i}{5}$ e) $\frac{4+5i}{3}$

2. Which of the following are true statements ?
The expression $(x^4 + y^4)$ is factorable into two or more non-constant polynomials with
(I) complex coefficients. (II) real coefficients.
(III) integral coefficients.

a) (I) only b) (I) and (II) only
c) (I) and (III) only d) (I) , (II) and (III)
e) None of these statements is true.

3. How many solutions does $\sin x = x^2$ have ?

a) 0 b) 1 c) 2 d) 3 e) More than 3

4. A pile of logs has 30 logs in the first layer, 29 in the second , 28 in the third and so on . The top layer contains 10 logs. Find the total number of logs in the pile.

a) 400 b) 405 c) 410 d) 415 e) 420

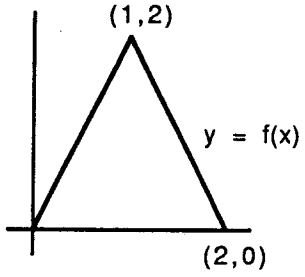
5. For what real values of x does $y^2 - x^3 + x = 0$ have a real solution y ?

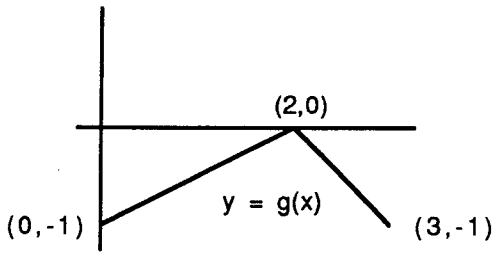
a) All values of x b) $x \geq 1$ or $x \leq 0$
c) $x \geq 1$ or $-1 \leq x \leq 0$ d) $x \leq -1$ or $0 \leq x \leq 1$ e) $x \geq -1$

6. How many perfect squares are there between 1000 and 2000 ?

a) 10 b) 13 c) 31 d) 44 e) None of these

7. Given the graphs of f and g, compute $g(f(g(f(1))))$.





a) -1 b) 0 c) 1 d) 2 e) 3

8. If A = { 1,2,.....,9 } then how many subsets of A of order 3 (that is each subset containing 3 elements) have "9" as one of their elements ?

a) 9 b) 28 c) 36 d) 56 e) 72

9. In how many ways can a rectangular cardboard (permanently fixed to a wall) with 5 congruent parallel strips,be painted by the 3 colors red,white and green in such a way that no two adjacent strips are painted by the same color ?

a) 10 b) 32 c) 48 d) 125 e) 243

10. If $f(x) = \sqrt{-x}$, then $f^{-1}(x)$, with its domain, is

a) $f^{-1}(x) = x^2$, $x \geq 0$ b) $f^{-1}(x) = -x^2$, $x \geq 0$
c) $f^{-1}(x) = x^2$, $x \leq 0$ d) $f^{-1}(x) = -x^2$, $x \leq 0$
e) The inverse does not exist.

11. The coefficient of the term $x^3 y^2$ in the expansion of $(2x - 3y)^5$ is

a) -720 b) -240 c) - 30 d) 240 e) 720

12. A right circular cylinder of radius r and height h is inscribed in a cone of altitude 12 and radius of base 4.The center of the base of the cylinder is at the center of the base of the cone. Then h as a function of r is equal to

a) $\frac{r}{3}$ b) 3r c) 3(4-r) d) 4(3-r) e) None of these

13. A rubber ball is dropped from a height of 10 meters. If it rebounds approximately one half the distance after each fall, then an approximation of the total distance the ball travels before coming to rest is

a) 10 b) 15 c) 20 d) 25 e) 30

14. Two perpendicular lines intersect at (2,1) . The y-intercept of one line expressed as a function of the y-intercept ,b, of the other line is

a) $\frac{b+3}{b-1}$ b) $\frac{2}{1-b}$ c) $\frac{2b}{b-1}$ d) $\frac{4-2b}{1-b}$ e) $\frac{5-b}{1-b}$

15. Suppose y is inversely proportional to the square of x. If the value of x is halved, then the value of y is

a) one-fourth of its former value b) half of its former value
c) double its former value d) four times its former value
e) indeterminable without more information

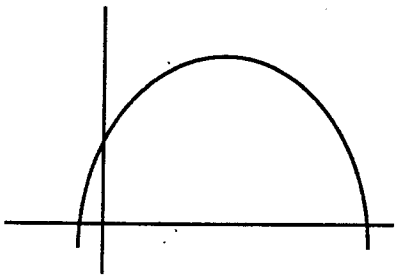
16. What is the ratio of the area of a square to the area of an equilateral triangle if they have the same perimeter ?

a) 1 b) $\frac{3\sqrt{3}}{4}$ c) $\frac{4\sqrt{3}}{3}$ d) $\frac{3\sqrt{3}}{2}$ e) None of these

17. Three machines A, B and C take 3 hours to do a job,when working together. Suppose A and B are identical and that C working alone can complete the entire job in 12 hours. How long would it take A working alone to do the job ?

a) 4 hours b) 8 hours c) 10 hours d) 12 hours e) none of these

18. If the graph of the quadratic $y = Ax^2 + Bx + C$ is pictured below, then which one of the following is true ?



a) $A > 0, B < 0, C < 0$

b) $A < 0, B < 0, C > 0$

c) $A > 0, B > 0, C > 0$

d) $A < 0, B > 0, C < 0$

e) $A < 0, B > 0, C > 0$

19. For which real values of x is $\sin(2x) < \sqrt{3}$?

a) $x < \frac{\sqrt{3}}{2}$ b) $-\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2}$ c) $x < \frac{\pi}{3}$

d) $-\frac{\pi}{3} < x < \frac{\pi}{3}$ e) All numbers

20. What is the remainder when 19^{90} is divided by 5 ?

a) 0 b) 1 c) 2 d) 3 e) 4

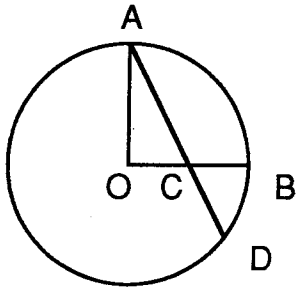
21. The Gastritic Cereal Corporation increased the net weight of cereal in its boxes by 20%, raised the price per box by 44%, and then offered a 25% discount off the new price. The overall effect of all these changes is that the price per unit weight decreased by

a) 1% b) 5% c) 10% d) 13.6% e) 15%

22. A solution to the equation $\log_5 8x + 2\log_5 x = 3$, is in the interval

a) $1 \le x < 2$ b) $2 \le x < 3$ c) $3 \le x < 4$ d) $4 \le x < 5$ e) $5 \le x < 6$

23. OA and OB are perpendicular radii of circle ABD, whose radius is 2. C is the midpoint of OB, and ACD is a straight line segment. How long is the line segment CD ?



a) $\frac{3}{\sqrt{5}}$ b) 1

c) $\frac{2}{\sqrt{3}}$ d) $\frac{\sqrt{5}}{2}$

e) $\sqrt{3}$

24. When I was in school , I never understood why the product of two negative numbers is positive. So, when I grew up and became second deputy assistant under-secretary of Education , I used my authority to decree that henceforth the product of two negative numbers would be negative (but its absolute value would still be as before), for example $(-2)(-3) = -6$. What law(s) did my decree violate ?

a) Commutative b) Associative c) Distributive
d) All of the preceding e) None of the preceding

25. A triangle has sides of lengths 12, 12 and 17 . A circle is drawn with the long side as diameter. Where , relative to the circle, is the vertex where the two short sides meet ?

a) Inside the circle b) On the circle
c) Outside the circle
d) It cannot be determined unless one of the angles of the triangle is also given
e) It cannot be determined even if one of the angles of the triangle is also given.

26. The radii of the two given circles are 4 and 2 , and the distance between their centers is 12. Find the length of the indicated common tangent segment AB .

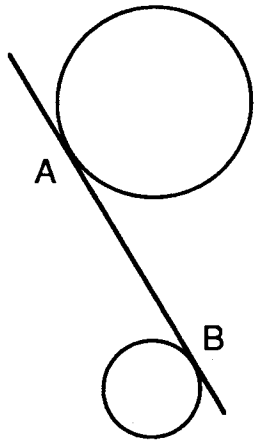
a) $4\sqrt{3}$

b) $6\sqrt{2}$

c) $5\sqrt{3}$

d) $6\sqrt{3}$

e) $7\sqrt{2}$



27. How many angles between 0^o and 360^o satisfy $\sin x + \cos x = -\sqrt{2}$?

a) Zero b) One c) Two d) Three e) Four

28. Let $f(x) = x^2$, $g(x) = x + 2$, $h(x) = x - 2$. Which of the following compositions results in shifting the graph of $y = f(x)$ so that the vertex is at $(-2,2)$?

a) $f(g(h(x)))$ b) $h(f(g(x)))$
c) $g(f(h(x)))$ d) $h(f(h(x)))$
e) $g(f(g(x)))$

29. The radius of a circle inscribed in a regular hexagon with side 20 is

a) 4 b) 5 c) 10 d) $20\frac{\sqrt{3}}{3}$ e) $10\sqrt{3}$

30. On which interval is $f(x) = \dfrac{1}{2\sqrt{\sqrt{x}-x} - 1}$, defined as a real-valued function ?

a) (-0.1 , 0.2) b) (0.2 , 0.5) c) (0.5 , 0.8)
d) (0.8 , 1.1) e) None of these

31. Three (ordinary and fair) dice are thrown. What is the probability of getting three (distinct) consecutive numbers ?

a) $\frac{1}{54}$ b) $\frac{1}{36}$ c) $\frac{1}{18}$ d) $\frac{1}{9}$ e) $\frac{1}{6}$

32. Suppose AC is a chord of a circle , with length equal to the radius of the circle. If B is any point of the minor arc AC , find the measure of the angle ABC.

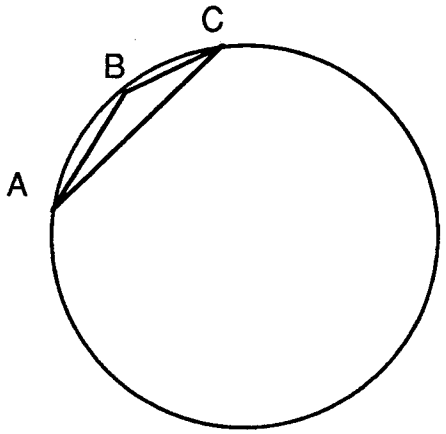
a) 120^o

b) 135^o

c) 145^o

d) 150^o

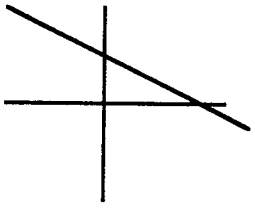
e) Not enough information is given

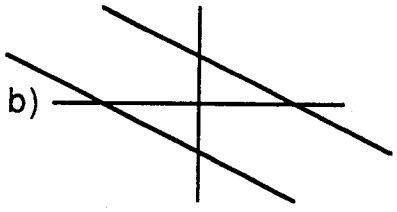


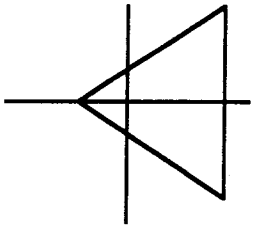
33. Start with a cube. Extend each of its faces to a whole plane. Into how many regions do these planes divide space ? (Regions are three dimensional, not parts of planes.)

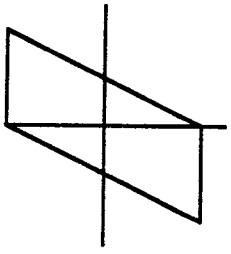
a) 18 b) 21 c) 24 d) 27 e) None of these

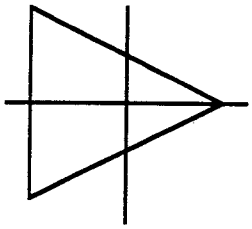
34. The graph of $|x + y| + |y| = 1$ looks most nearly like

a) 

b) 

c) 

d) 

e) 

35. Find all values of x for which $|\log_4 |x|| < 1$.

a) $0 < x < 4$ or $-4 < x < 0$ b) $\frac{1}{4} < x < 4$ or $-4 < x < -\frac{1}{4}$ c) $-4 < x < 4$

d) $\frac{1}{4} < x < 4$ e) $0 < x < 4$

36. If $\pi < x < \frac{3\pi}{2}$, then Arcsin (sinx) is equal to

a) x b) $x - \pi$ c) $\pi - x$ d) $2\pi - x$ e) $x + \frac{\pi}{2}$

37. The integers k and m are each randomly selected from the integers 0 through 4. What is the probability that the equation $x^2 + kx + m = 0$ has complex roots (with non-zero imaginary part) ?

a) $\frac{2}{5}$ b) $\frac{11}{25}$ c) $\frac{12}{25}$ d) $\frac{13}{25}$ e) $\frac{14}{25}$

38. If $x + y = 6$ and $xy = 7$ then the expression $x^3 + y^3$ is equal to

a) 90 b) 126 c) 150 d) 195 e) 216

39. Suppose that the following three statements are assumed to be true :
(I) All people who eat oat bran cereal are healthy people.
(II) Some people don't eat cereal .
(III) All people are either healthy or unhealthy but not both.
Which one of the following statements logically follows from the preceding ?

a) All healthy people who eat cereal,eat oat bran cereal .
b) All people who don't eat oat bran cereal are unhealthy.
c) Some healthy people don't eat cereal .
d) All unhealthy people who eat cereal ,don't eat oat bran cereal .
e) Some people who eat cereal but not oat bran are unhealthy .

40. A car travels at a speed of 36 kilometers per hour for the first 30 kilometers . How far would the car have to travel at a speed of 60 km/hr to have an average speed of 48 km/hr for the whole trip ?

a) 42 km b) 44 km c) 46 km d) 48 km e) 50 km

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