# THIRTY-SECOND ANNUAL
# MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

## PART I

October 12, 1988

## INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. **Check to insure that your six-digit code number has been recorded correctly.** Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please quit working immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine an answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "**e)** none of these" or "**e)** none of the above". If you believe none of the first four alternatives to be correct, mark **e)** in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You may now open the test booklet and begin.

1.– If the population of a town increases by 25% one year, by 30% the next year, and by 35% in a third year, then the percent increase in the population for the three year period is nearest to

a) 90% b) 100% c) 120% d) 220% e) 390%

2.– A girl 5 feet 6 inches tall casts a shadow 2 feet long, while a nearby tree casts a 22 foot shadow. How tall is the tree?

a) 8 feet b) 44 feet c) $60\frac{1}{2}$ feet d) 66 feet e) none of the above

3.– Two cars are travelling along the same highway at constant speeds. At 1:00 PM they are 112 miles apart and at 2:00 PM the distance has decreased to 80 miles. When will the cars meet or pass?

a) 4:45 PM b) 4:30 PM c) 3:45 PM d) 3:30 PM e) none of the above

4.– The area of the trapezoid with vertices at (0,3), (3,0), (0,9), and (9,0) is

a) $18\sqrt{2}$ b) 36 c) 72 d) $36\sqrt{2}$ e) none of the above

5.– What is the solution set of $|x| > x+1$ , where $|x|$ denotes absolute value?

a) $\{x|x < -\frac{1}{2}\}$
b) $\{x| -\frac{1}{2} < x < 0\}$
c) $\{x|x > -\frac{1}{2}\}$
d) $\{x|x < 0\}$
e) the empty set

6.– How many angles $\theta$ between 0 degrees and 360 degrees satisfy the equation $\cos\theta - \tan\theta = 0$ ?

a) 0 b) 1 c) 2 d) 3 e) 4

7.– The expression $\frac{1}{x} - \frac{2}{y}$ is equal to

a) $\frac{y-2x}{xy}$
b) $\frac{-1}{x-y}$
c) $\frac{y-2x}{x+y}$
d) $\frac{-1}{xy}$
e) none of the above

8.– On what interval is $(x-1)(x+2) < 0$ ?

a) $-1 < x < 2$
b) $1 < x < 2$
c) $-2 < x < 1$
d) $-2 < x < 0$
e) $x < 0$

9.– If A, B and C are the three angles in a right triangle, then $\cos^2 A + \cos^2 B + \cos^2 C$ is always equal to

a) 2
b) $2\cos^2 B$
c) $2(cos^2 B + cos^2 C)$
d) $2cos^2(B+C)$
e) none of the above

10.– The number $\log_{10} 10000$ is equal to

a) 2 b) 3 c) 4 d) 5 e) none of the above

11.– The polynomial $f(x)$ has 6 non-zero coefficients, and the sum of all the coefficients is 0. One factor of the polynomial $f(x)$ is:

a) $x$ b) $x+1$ c) $x-1$ d) $x^2+1$
e) This is not enough information to show that any of the above are factors of $f(x)$

12.– A circle of radius 4 cm. is inscribed in an equilateral triangle. What is the area of the triangle?

a) $16\sqrt{3}$ b) $24\sqrt{3}$ c) $48\sqrt{3}$ d) $64\sqrt{3}$ e) none of the above

13.– Karen rolls a fair six-sided die six times. What is the probability that each face will occur (come up on top) exactly once? (Note: $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, read "six factorial" )

a) $\frac{1}{6}$
b) $6(\frac{1}{6})$
c) $(\frac{1}{6})^6$
d) $6(\frac{1}{6!})$
e) $(\frac{1}{6})^6 \cdot 6!$

14.– A standard deck of cards contains 52 cards, of which half are red (diamonds or hearts) and half are black (spades or clubs). You have a well shuffled deck of cards and draw 2 at a time. If both cards are red, put them in a red pile on the left side of your table. If both cards are black, put them in the black pile on the right side of your table. When you draw a pair of mixed color, throw them both in the trash can. What is the probability that you will have exactly the same number of cards in each of the two piles on the table when you come to the end of the deck?

a) 1 b) 0 c) 1/52 d) $1/2^{26}$ e) $26!/52!$

15.– The expression

$$\frac{i}{2+3i}$$

(where $i$ is a square root of -1) is equal to:

a) $\frac{2i+3}{13}$
b) $-2i-3$
c) $-\frac{2i+3}{5}$
d) $\frac{2i-3}{13}$
e) $\frac{3-2i}{5}$

16.– If $2\log_y N = \log_x N$ for all $N > 0$, then we can conclude that

a) $y = 2x$
b) $y = x/2$
c) $y = \sqrt{x}$
d) there are no values of $x$ and $y$ such that the equation holds for all $N > 0$
e) none of the above

17.– One thousand perfect cubical blocks of the same size are stacked in the middle of the floor as a larger (10 by 10 by 10) cube. How many of the original blocks are hidden from view?

a) 271 b) 500 c) 512 d) 576 e) 729

18.– For what value of $x$ is $9^{2x} = 27^{3x+1}$ ?

a) -3/5 b) 3/2 c) 4/9 d) -1 e) 3/5

19.– One solution to the equation

$$x = \frac{1}{1+\frac{1}{x}}$$

is:

a) 0 b) 1 c) $\frac{1}{2}$

d) there is no solution to this equation

e) none of the above

20.– The numbers $p$ and $q$ are said to be twin primes if $p$ and $q$ are both prime numbers and $|p-q| = 2$. How many sets of twin primes are there with both primes less than 50?

a) 4 b) 5 c) 6 d) 7 e) 8

21.– Define a function $f$ on the positive integers by

$$f(n) = \begin{cases} 3n+1, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd} \end{cases}$$

What is $f(f(3))$ ?

a) 12 b) 19 c) 20 d) 31 e) none of the above

22.– What is the smallest positive angle $\theta$ satisfying the equation $\cos(3\theta) + \sin(3\theta) = 0$ ?

a) 0 degrees

b) 30 degrees

c) 45 degrees

d) 135 degrees

e) none of the above is the smallest solution (or there is no solution)

23.– A square is inscribed in the ellipse $x^2 + 4y^2 = 4$, with two sides parallel to the $x$-axis. What is the area of the square?

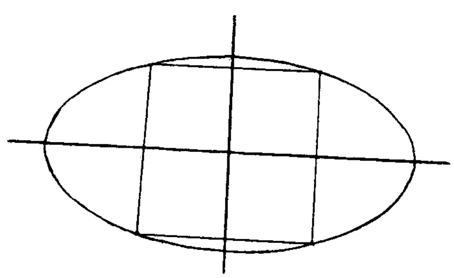
a) 4/5

b) 5

c) 16/5

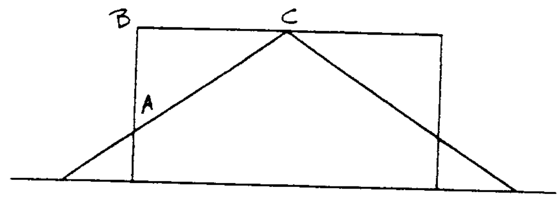
d) 8/5

e) none of the above



24.- Given the rectangle and isosceles triangle as shown with AB equal to 2/3 of the height of the rectangle, and C at the midpoint of the top side of the rectangle, then the area of the triangle $A_T$ is related to the area of the rectangle $A_R$ by

a) $A_T = (1/2)A_R$
b) $A_T = (2/3)A_R$
c) $A_T = (3/4)A_R$
d) $A_T = (5/6)A_R$
e) none of the above




25.- Which of the following is a factor of $2^{2001} + 1$ ?

a) $2^3 + 1$ b) $2^3 - 1$ c) $2^{667} + 1$ d) $2^{667} - 1$
e) more than one of the above

26.- A triangle is inscribed in a circle of radius 1. One side of the triangle is a diameter of the circle, and another side has length 1. What is the area of this triangle?

a) $\frac{\sqrt{3}}{2}$ b) 1 c) 2 d) $\sqrt{3}$
e) none of the above

27.- The inequality $1 < |x - 2| < 4$ is equivalent to

a) $3 < x < 6$
b) $-2 < x < 1$ or $3 < x < 6$
c) $-2 < x < 0$ or $2 < x < 6$
d) $-2 < x < 0$ and $2 < x < 6$
e) none of the above

28.- Given the following true statement: "Jerry will not run in the race only if his leg is not better", we can conclude that:

a) If Jerry's leg is better he may run in the race
b) If Jerry's leg is better he will run in the race
c) If Jerry runs in the race then his leg will be better
d) If Jerry does not run in the race then his leg may not be better
e) If Jerry's leg is not better then he will not run in the race

29.- A rectangular pyramid of volume $V$ is cut by a plane parallel to the base, into a smaller pyramid and a truncated pyramid. The height of the plane (and of the bottom part or truncated pyramid) is 1/3 of the height of the original (larger) pyramid. The volume of the bottom part is

a) $(5/9)V$
b) $(19/27)V$
c) $(8/27)V$
d) $(1/3)V$
e) none of the above

30.- A professor strolled leisurely away from home at 2.5 miles per hour. Suddenly realizing he was late for an appointment, he turned around and ran home at 7.5 miles per hour by the same route. What was his average speed for the whole jaunt?

a) 3 Miles per hour
b) 3.75 Miles per hour
c) 4 Miles per hour
d) 5 Miles per hour
e) 6 Miles per hour

31.- Which of the following lines intersect the line $y = x + 3$ in the third quadrant?

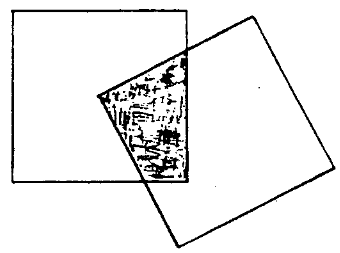
a) $y = (\frac{1}{2})x + 4$
b) $y = -x - 2$
c) $y = -x + 2$
d) $y = 1$
e) $y = -(\frac{1}{2})x - 4$

32.- Which of the following numbers is nearest to the smaller positive solution to the equation $x^2 - 1429x + 1 = 0$ ?

a) .7 b) .07 c) .007 d) .0007 e) .00007

33.- Two squares both with sides of length 4 are positioned so that the center of one square is at a corner of the other square. What is the area of the overlap?

a) $2\sqrt{2}$
b) 4
c) $3\sqrt{2}$
d) There is insufficient information to determine the area of the overlap.
e) none of the above



34.- Tom's mother usually picks him up when school lets out at 3:00 PM. When he left school at 3:00 PM on Tuesday, she was not there so he began walking home at a rate of 3 miles per hour. After 24 minutes of walking he met his mother who drove him the rest of the way home. They arrived home 22 minutes later than usual. What is his mother's average driving speed?

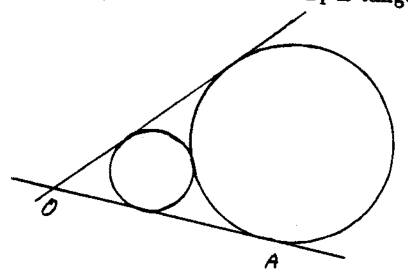
a) 24 miles per hour
b) 27 miles per hour
c) 30 miles per hour
d) 36 miles per hour
e) none of the above

35.- The symbols $37_r$ and $47_s$ represent two digit numbers in the bases $r$ and $s$ respectively, as the subscript 10 would mean a number in the usual decimal notation. If $37_r = 47_s$ then the smallest possible respective values of $r$ and $s$ on the following list are:

a) $r = 16$ , $s = 12$
b) $r = 12$ , $s = 16$
c) $r = 10$ , $s = 8$
d) $r = 8$ , $s = 10$
e) none of the above are possible values.

36.– Two circles with radii 8 and 2 respectively are tangent to each other. Two lines $L_1$ and $L_2$ are each tangent to both circles, and intersect each other at the point O . Call the point at which line $L_1$ is tangent to the larger circle A . What is the distance from A to O ?

a) 6
b) 8
c) 8/3
d) 32/3
e) none of the above




37.– The length of one base of a trapezoid, originally $b_1$, is increased by 3 to $b_1 + 3$, while the altitude is held fixed. How much must be added to the other base, originally $b_2$, so that the area of the new trapezoid will be twice the area of the original trapezoid?

a) $b_1 + b_2 - 3$
b) 3
c) $b_1 + 3$
d) $b_1 + b_2 + 3$
e) the amount to add would depend on the altitude or height

38.– Determine the constant $a > 0$ such that the graphs of the two curves

$$y = \frac{2}{x} \quad \text{and} \quad x^2 + y^2 = a^2$$

intersect in exactly two points.

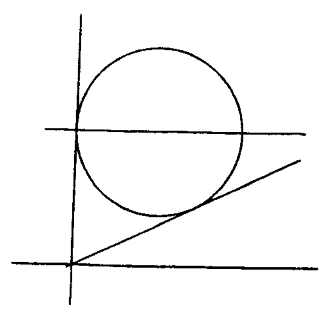
a) 1 b) $\sqrt{2}$ c) 3 d) 2 e) none of the above

39.– If a square and a circle have the same perimeter, then the ratio of the area of the square to the area of the circle is

a) $\pi$ b) $\pi/2$ c) $\pi/4$ d) $2\pi$
e) none of the above

40.– A circle entirely in the first quadrant is tangent to the $y$-axis and to the line $y = x/2$, and the center of the circle is on the line $y = 2$. Find the radius of the circle.

a) $\frac{4}{3}$
b) $-1 + \sqrt{5}$
c) $\frac{-1 + \sqrt{5}}{2}$
d) $1 + \sqrt{5}$
e) $\frac{1 + \sqrt{5}}{2}$



The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America

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