THIRTY-FIRST ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

PART II

December 9, 1987

1. Let $D(n)$ denote the number of positive factors of the integer n. For example, $D(6) = 4$, since the factors of 6 are 1, 2, 3, and 6. Note that $D(n) = 2$ if and only if n is a prime number.
   (3 pts) (a) Describe the set of all solutions to the equation $D(n) = 5$.
   (3 pts) (b) Describe the set of all solutions to the equation $D(n) = 6$.
   (4 pts) (c) Find the smallest n such that $D(n) = 21$.

If the factorisation of n into its distinct prime factors is $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then

$$D(n) = (e_1+1)(e_2+1)\cdots(e_k+1),$$

since specifying a divisor of n amounts to specifying an exponent from 0 to e_i for each p_i.

(a) Since 5 is prime, the only way that $D(n)$ can equal 5 is for $k = 1$ and $e_1 = 4$ with the notation above. Thus the solution consists of all fourth powers of prime numbers: 2^4, 3^4, 5^4, etc.

(b) Arguing as in part (a) and observing that the only factorisations of 6 are 6 and $2 \cdot 3$, we see that $D(n) = 6$ if and only if either $k = 1$ and $e_1 = 5$, or $k = 2$ and $\{e_1, e_2\} = \{1, 2\}$. Thus the solutions are precisely those numbers of the form p^5 or of the form $p^2 q$, where p and q are distinct primes.

(c) As in part (b), the solutions to $D(n) = 21$ are $n = p^{20}$ and $n = p^2 q^6$, for distinct primes p and q. To find the smallest such number, we need only compare 2^{20} and $3^2 2^6$, since any other choices for p and q clearly result in a larger n. But since $3^2 < 2^{14}$, the latter is smaller. Thus the answer is $n = 3^2 2^6 = 9 \cdot 64 = 576$.

2. At a party with n married couples present (and no one else), various people shook hands with various other people. Assume that no one shook hands with his or her spouse, and no one shook hands with the same person more than once. At the end of the evening Mr. Jones asked everyone else, including his wife, how many hands he or she had shaken. To his surprise, he got a different answer from each person. Determine the number of hands that Mr. Jones shook that evening,
   (2 pts) (a) if $n = 2$.
   (3 pts) (b) if $n = 3$.
   (5 pts) (c) if n is an arbitrary positive integer (the answer may depend on n).

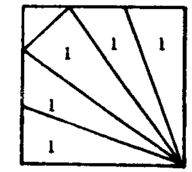
We will show that Mr. Jones shook $n-1$ hands for part (c), from which it follows that the answers to parts (a) and (b) are 1 and 2, respectively. (The approach given here easily specialises to simpler arguments for these parts.)

There are $2n$ people at the party. Since each person shook hands with at least 0 and at most $2n-2$ people (self and spouse are excluded), the $2n-1$ distinct answers that Mr. Jones received must be precisely the set of numbers $\{0, 1, 2, \ldots, 2n-2\}$. Now the person who shook hands with $2n-2$ people—let's call him Mr. A without loss of generality—shook hands with everyone except Mrs. A, so only Mrs. A can be the person who shook hands with nobody. Note in particular that A $\neq$ Jones.

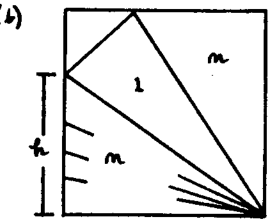
Next, without loss of generality, assume that Mr. B is the person who shook hands with $2n-3$ people. Since he did not shake Mrs. A's hand, he must have shaken hands with everyone else except his wife. This means that everyone not yet mentioned shook at least two hands (Mr. B's and Mr. A's), so the person who shook only one hand must have been Mrs. B (she shook only Mr. A's hand). Again, B $\neq$ Jones.

Continuing in this way, we find Mr. and Mrs. C, who shook $2n-4$ and 2 hands, respectively, and so on, until finally we find Mr. and Mrs. X who shook n and $n-2$ hands, respectively. Note that in particular Mr. Jones (and also Mrs. Jones, for that matter) must have shaken the hand of every other man and no other woman (under our WOLOG assumptions), so the answer is $n-1$.

3. Let n be a positive integer. A square is divided into triangles in the following way. A line is drawn from one corner of the square to each of n points along each of the opposite two sides, forming $2n+2$ nonoverlapping triangles, one of which has a vertex at the opposite corner and the other $2n+1$ of which have a vertex at the original corner. The figure shows the situation for $n = 2$. Assume that each of the $2n+1$ triangles with a vertex in the original corner has area 1. Determine the area of the square,
   (4 pts) (a) if $n = 1$.
   (6 pts) (b) if n is an arbitrary positive integer (the answer may depend on n).



(b)



LET x = EDGELENGTH, $x^2 = A$,

THEN $\frac{1}{2} h x = n$, SO $h = \frac{2n}{x}$,

AND TRIANGLE IN UPPER LEFT CORNER HAS LEGS OF LENGTH $x - \frac{2n}{x}$.

WE CAN NEXT EXPRESS THE AREA OF THE SQUARE AS THE SUM OF THE AREAS OF THE TRIANGLES:

$$A = 2n + 1 + \frac{1}{2}\left(x - \frac{2n}{x}\right)^2$$

$$= 2n + 1 + \frac{1}{2}\left(\frac{x^2 - 2n}{x}\right)^2$$

OR $$A = 2n+1+\frac{1}{2}\frac{(A-2n)^2}{A}$$

SIMPLIFYING,

$$A^2 - 2A - 4n^2 = 0, \quad \text{SO} \quad A = 1 \pm \sqrt{1+4n^2}$$

THE POSITIVE ROOT IS $1 + \sqrt{1+4n^2}$.

(a) WHEN $n = 1$, $A = 1 + \sqrt{5}$.

(Another approach is to concentrate on the length of the long altitude of the middle triangle.)

4. Arthur and Betty play a game with the following rules. Initially there are one or more piles of stones, each pile containing one or more stones. A legal move consists either of removing one or more stones from one of the piles, or, if there are at least two piles, combining two piles into one (but not removing any stones). Arthur goes first, and play alternates until a player cannot make a legal move; the player who cannot move loses.

(2 pts) (a) Determine who will win the game if initially there are two piles, each with one stone, assuming that both players play optimally.

(4 pts) (b) Determine who will win the game if initially there are two piles, each with n stones, assuming that both players play optimally; n is a positive integer, and the answer may depend on n.

(4 pts) (c) Determine who will win the game if initially there are n piles, each with one stone, assuming that both players play optimally; n is a positive integer, and the answer may depend on n.

Clearly the winner is the person who takes the last stone.

(a) Betty can win with optimal play. Whatever Arthur does first, there will be one pile remaining for Betty, and she can remove all of it to win.

(b) Again Betty will win with optimal play, regardless of the value of n. If Arthur ever merges the two piles into one, then Betty can win by taking the entire pile, so we can assume that Arthur never makes that move and instead always removes one or more stones from one of the piles. Betty's strategy is simply to copy Arthur's move in the other pile—if he removes k stones from one pile, she removes k stones from the other. By doing so, she will always present Arthur again with two piles of equal size (unless she has just won by removing the last stone), so by induction she will always win (the base case is part (a)).

(c) We claim that Arthur wins if n is odd and Betty wins if n is even. The winner's strategy is simply to remove one pile at each turn, the pile with two stones in it if the opponent has created one on the previous move. We will prove by induction that with optimal play a player looking at an even number of piles of one stone each will lose and a player looking an at odd number of piles of one stone each will win. This is clearly true for $n = 1$. Assume the inductive hypothesis and suppose a player is facing $n > 1$ piles of one stone each. If n is odd, then by removing one stone he can present his opponent with the losing position (by the inductive hypothesis) of an even number $(n-1)$ of piles of one stone each. On the other hand, assume n is even. If the player removes one stone, then the opponent is looking at an odd number of piles and so wins by the inductive hypothesis. If the player combines two piles into one, then the opponent takes this pile, either winning outright (if there are no more stones) or else leaving the original player facing $n-2$ (an even number) of piles of one stone each, which is a losing position by the inductive hypothesis.

5. Suppose x and y are real numbers such that $0 < x < y$. Define a sequence $A_0, A_1, A_2, A_3, \ldots$, by setting $A_0 = x$, $A_1 = y$, and then $A_n = |A_{n-1}| - A_{n-2}$ for each $n \geq 2$ (recall that $|A_{n-1}|$ means the absolute value of A_{n-1}).

(4 pts) (a) Find all possible values for A_6 in terms of x and y.

(6 pts) (b) Find values of x and y so that $A_{1987} = 1987$ and $A_{1988} = -1988$ (simultaneously).

A_0	x	$(+)$		
A_1	y	$(+)$		
A_2	$y-x$	$(+)$		
A_3	$-x$	$(-)$		
A_4	$2x-y$,	TWO CASES NOW NEEDED:		
	$2x-y \geq 0$		$2x-y < 0$	
A_5	$3x-y$	$(+)$	$y-x$	$(+)$
A_6	x		$2y-3x$	$(+)$
A_7	$y-2x$	$(-)$	$y-2x$	$(+)$
A_8	$x-y$	$(-)$	$x-y$	$(-)$
A_9	x		x	
A_{10}	y		y	

(a) THE POSSIBLE VALUES OF A_6 ARE x, $2y-3x$.

(b) CONTINUING THE CALCULATION TO A_9, A_{10}, WE NOTE PERIODICITY :

$$A_i = A_{9k+i}.$$

THUS $A_{1987} = A_7$, $A_{1988} = A_8$.

FROM THE ABOVE

$$\left.\begin{aligned} A_7 &= y-2x = 1987 \\ A_8 &= x-y = -1988 \end{aligned}\right\}$$

SOLVE :

$x = 1$

$y = 1989$