4. Arthur and Betty play a game with the following rules. Initially there are one or more piles of stones, each pile containing one or more stones. A legal move consists either of removing one or more stones from one of the piles, or, if there are at least two piles, combining two piles into one (but not removing any stones). Arthur goes first, and play alternates until a player cannot make a legal move; the player who cannot move loses.

(2 pts) (a) Determine who will win the game if initially there are two piles, each with one stone, assuming that both players play optimally.
(b) Determine who will win the game if initially there are two piles, each with \( n \) stones, assuming that both players play optimally; \( n \) is a positive integer, and the answer may depend on \( n \).
(c) Determine who will win the game if initially there are \( n \) piles, each with one stone, assuming that both players play optimally; \( n \) is a positive integer, and the answer may depend on \( n \).

Clearly the winner is the person who takes the last stone.

(a) Betty can win with optimal play. Whatever Arthur does first, there will be one pile remaining for Betty, and she can remove all of it to win.

(b) Again Betty will win with optimal play, regardless of the value of \( n \). If Arthur ever merges the two piles into one, then Betty can win by taking the entire pile, so we can assume that Arthur never makes that move and instead always removes one or more stones from one of the piles. Betty's strategy is simply to copy Arthur's move in the other pile—if he removes \( k \) stones from one pile, she removes \( k \) stones from the other. By doing so, she will always present Arthur again with two piles of equal size (unless she has just won by removing the last stone), so by induction she will always win (the base case is part (a)).

(c) We claim that Arthur wins if \( n \) is odd and Betty wins if \( n \) is even. The winner's strategy is simply to remove one pile at each turn, the pile with two stones in it if the opponent has created one on the previous move. We will prove by induction that with optimal play a player looking at an even number of piles of one stone each will lose and a player looking at an odd number of piles of one stone each will win. This is clearly true for \( n = 1 \). Assume the inductive hypothesis and suppose a player is facing \( n > 1 \) piles of one stone each. If \( n \) is odd, then by removing one stone he can present his opponent with the losing position (by the inductive hypothesis) of an even number (\( n - 1 \)) of piles of one stone each. On the other hand, assume \( n \) is even. If the player removes one stone, then the opponent is looking at an odd number of piles and so wins by the inductive hypothesis. If the player combines two piles into one, then the opponent takes this pile, either winning outright (if there are no more stones) or else leaving the original player facing \( n - 2 \) (an even number) of piles of one stone each, which is a losing position by the inductive hypothesis.

5. Suppose \( x \) and \( y \) are real numbers such that \( 0 < x < y \). Define a sequence \( A_0, A_1, A_2, A_3, \ldots \) by setting \( A_0 = x \), \( A_1 = y \), and then \( A_n = |A_{n-1} - A_{n-2}| \) for each \( n \geq 2 \) (recall that \( |A_{n-1}| \) means the absolute value of \( A_{n-1} \)).

(1 pts) (a) Find all possible values for \( A_4 \) in terms of \( x \) and \( y \).
(6 pts) (b) Find values of \( x \) and \( y \) so that \( A_{1987} = 1987 \) and \( A_{1988} = -1988 \) (simultaneously).

\[ \begin{align*}
A_0 & = x \\
A_1 & = y \\
A_2 & = y - x \\
A_3 & = -x \\
A_4 & = 2x - y, \quad \text{Two cases now needed:} \\
\end{align*} \]

\[ \begin{align*}
& 2x - y \geq 0 \\
& 2x - y < 0 \\
A_5 & = 3y - 2x \\
A_6 & = y - 2x \\
A_7 & = y - 2x \\
A_8 & = x \\
A_9 & = y \\
\end{align*} \]

(a) The possible values of \( A_6 \) are \( x \), \( 2y - 3x \).

(b) Continuing the calculation to \( A_9 \), \( A_{10} \), we note periodicity: \( A_{i+1} = A_{9i+1} \).

Thus \( A_{1987} = A_7 \), \( A_{1988} = A_8 \).

From the above
\[ \begin{align*}
A_7 & = y - 2x = 1987 \quad \text{Solve:} \quad \begin{cases} x = 1 \\
y = 1987 \end{cases} \\
A_8 & = x - y = -1988 \quad \text{Solve:} \quad y = 1989
\end{align*} \]