MMPC CODE NUMBER

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THIRTY-FIRST ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

PART II

December 9, 1987

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Carefully record your six-digit MMPC code number in the upper right-hand corner of this page. This is the only way to identify you with this test booklet. **PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.**
2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions. To receive full credit for a problem, you are expected to justify your answer.
3. You are not expected to solve all problems completely. Look over all the problems and work first on those that interest you the most.
4. Each problem is on a separate page. You should show most of your work on that page. If it is necessary to use additional paper for your answer, indicate this on the exam page and write your identification number and the problem number in the upper right-hand corner of each additional sheet.
5. If you are unable to completely solve a particular problem, partial credit may be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.
6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate ideas of how to attack the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.
7. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference material, computational aids, or any other aid is likewise prohibited. When the supervisor announces that the 100 minutes have elapsed, please cease work immediately and insert all significant extra paper into the test booklet. It is not necessary to return scratch paper on which routine numerical calculations have been made.
8. You may now open the test booklet and begin.

Score :

1	2	3	4	5	TOTAL

1. Let $D(n)$ denote the number of positive factors of the integer n. For example, $D(6) = 4$, since the factors of 6 are 1, 2, 3, and 6. Note that $D(n) = 2$ if and only if n is a prime number.

   (3 pts) (a) Describe the set of all solutions to the equation $D(n) = 5$.

   (3 pts) (b) Describe the set of all solutions to the equation $D(n) = 6$.

   (4 pts) (c) Find the smallest n such that $D(n) = 21$.

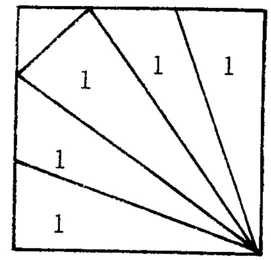
2. At a party with n married couples present (and no one else), various people shook hands with various other people. Assume that no one shook hands with his or her spouse, and no one shook hands with the same person more than once. At the end of the evening Mr. Jones asked everyone else, including his wife, how many hands he or she had shaken. To his surprise, he got a different answer from each person. Determine the number of hands that Mr. Jones shook that evening,

(2 pts) (a) if $n = 2$.

(3 pts) (b) if $n = 3$.

(5 pts) (c) if n is an arbitrary positive integer (the answer may depend on n).

3. Let n be a positive integer. A square is divided into triangles in the following way. A line is drawn from one corner of the square to each of n points along each of the opposite two sides, forming $2n+2$ nonoverlapping triangles, one of which has a vertex at the opposite corner and the other $2n+1$ of which have a vertex at the original corner. The figure shows the situation for $n=2$. Assume that each of the $2n+1$ triangles with a vertex in the original corner has area 1. Determine the area of the square,



(4 pts) (a) if $n=1$.

(6 pts) (b) if n is an arbitrary positive integer (the answer may depend on n).

4. Arthur and Betty play a game with the following rules. Initially there are one or more piles of stones, each pile containing one or more stones. A legal move consists either of removing one or more stones from one of the piles, or, if there are at least two piles, combining two piles into one (but not removing any stones). Arthur goes first, and play alternates until a player cannot make a legal move; the player who cannot move loses.

(2 pts) (a) Determine who will win the game if initially there are two piles, each with one stone, assuming that both players play optimally.

(4 pts) (b) Determine who will win the game if initially there are two piles, each with n stones, assuming that both players play optimally; n is a positive integer, and the answer may depend on n.

(4 pts) (c) Determine who will win the game if initially there are n piles, each with one stone, assuming that both players play optimally; n is a positive integer, and the answer may depend on n.

5. Suppose x and y are real numbers such that $0 < x < y$. Define a sequence A_0, A_1, A_2, A_3, ..., by setting $A_0 = x$, $A_1 = y$, and then $A_n = |A_{n-1}| - A_{n-2}$ for each $n \geq 2$ (recall that $|A_{n-1}|$ means the absolute value of A_{n-1}).

(4 pts) (a) Find all possible values for A_6 in terms of x and y.

(6 pts) (b) Find values of x and y so that $A_{1987} = 1987$ and $A_{1988} = -1988$ (simultaneously).

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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Michigan State University

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ACKNOWLEDGEMENTS

The following corporations and professional organizations have contributed generously to this competition.

Avco Container Corporation
Ford Motor Company
Kuhlman Corporation
Michigan Bell Telephone
Michigan Council of Teachers of Mathematics
Unisys Corporation
The Upjohn Company

The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.

O-17239